23. The $W^{k,p}$-continuity of Wave Operators for Schrödinger Operators

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(Communicated by Kiyosi ITÔ, M. J. A., April 12, 1993)

1. Introduction. Theorems. For Schrödinger operators $H = H_0 + V(x)$ and $H_0 = D_1^2 + \cdots + D_m^2$, $D_j = -i\partial/\partial x_j$, the wave operators $W_\pm$ and $Z_\pm$ are defined by the limits in $L^2 \equiv L^2(R^m)$:

$$(1.1) \qquad W_\pm u = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}u, \quad Z_\pm u = \lim_{t\to\pm\infty} e^{itH_0}e^{-itH}P_c(H)u,$$

where $P_c(H)$ is the orthogonal projection onto the continuous spectral subspace $L_c^2(H)$ for H. We assume that $V(x)$ satisfies the following condition, where $m_* = (m-1)/(m-2)$, $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\mathscr{F}$ is the Fourier transform. We take and fix $\sigma > 2/m_*$, $\delta > \max(m+2, 3m/2-2)$ and an integer $l \geq 0$.

Assumption 1.1. *$V(x)$ is a real valued function on R^m, $m \geq 3$, such that $\mathscr{F}(\langle x\rangle^\sigma D_x^\alpha V) \in L^{m_*}$ for any $|\alpha| \leq l$ and satisfies either* (1) $\|\mathscr{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}} \equiv C(V)$ *is sufficiently small or* (2) $m = 2m'-1$ *is odd and* $|D^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\delta}$ *for any* $|\alpha| \leq \max\{l, m'-4+l\}$.

Under the assumption, V is H_0-bounded and is short-range in the sense of Agmon [1]. Hence H with domain $D(H) = D(H_0) = W^{2,2}$ is selfadjoint and both limits in (1.1) exist ([1], [8]); $W_\pm$ are partial isometries from L^2 onto $L_c^2(H)$ and $Z_\pm = W_\pm^*$. Consequently, the continuous part H_c of H is unitarily equivalent to H_0 and, for any Borel function f, $f(H)P_c(H) = W_\pm f(H_0)W_\pm^*$, $f(H_0) = W_\pm^* f(H)\, P_c(H)W_\pm$. The main result of this paper is the following

Theorem 1.1. *Let V satisfy Assumption 1.1 and let 0 be neither eigenvalue nor resonance of H. Then, for any $1 \leq p \leq \infty$ and integral $0 \leq k \leq l$, $W_\pm$ and $Z_\pm$ originally defined on $L^2 \cap W^{k,p}$ can be extended to bounded operators in $W^{k,p}$.*

Remark 1.1. We say 0 is resonance of H if $-\Delta u(x) + V(x)u(x) = 0$ has a solution u such that $\langle x\rangle^{-\gamma}u(x) \in L^2$ for any $\gamma > 1/2$ but not for $\gamma = 0$. Under the assumption, 0 is not resonance if $m \geq 5$, and is neither eigenvalue nor resonance if $C(V)$ is small enough.

Remark 1.2. If 0 is resonance, Theorem 1.1 never holds. If 0 is eigenvalue of H, then it does not hold in general. This can be seen by comparing the results of [3] or [9] with Theorem 1.3 below.

In the sequel, we always assume that the condition of Theorem 1.1 is satisfied. For Banach spaces X and Y, $B(X, Y)$ is the space of bounded operators from X to Y, $B(X) = B(X, X)$. Theorem 1.1 yields the following

Theorem 1.2. *Let $1 \leq p, q \leq \infty$ and let $0 \leq k, k' \leq l$ be integers. Then*:

$$C^{-1}\|f(H_0)\|_{B(W^{k,p},W^{k',q})} \leq \|f(H)P_c(H)\|_{B(W^{k,p},W^{k',q})} \leq C\|f(H_0)\|_{B(W^{k,p},W^{k',q})},$$

where the constant $C > 0$ is independent of Borel functions f.

An immediate corollary of Theorem 1.2 is the $L^p - L^q$ estimate for time dependent Schrödinger, Klein-Gordon and wave equations. Under slightly different conditions on V, such estimate has been proven recently for Schrödinger and wave equations ([5], [2]). See [4] for related results.

Theorem 1.3. *Let $0 \le k \le l$ be integral, $2 \le p \le \infty$ and $1/p + 1/q = 1$. Then:*

$$\| e^{-itH} P_c(H) u \|_{W^{k,p}} \le C_{kp} | t |^{m(1/p-1/2)} \| u \|_{W^{k,q}}, \quad u \in L^2 \cap W^{k,q}.$$

Theorem 1.4. *Let $0 \le k \le l$ be integral, $2 \le p \le 2(m+1)/(m-1)$ and $1/p + 1/q = 1$. Then, there exists a constant $C_{kp} > 0$ such that for any $\phi, \psi \in L_c^2(H) \cap W^{k,q}$ the solution $u(t, x)$ of the Cauchy problem $\partial^2 u/\partial t^2 = \Delta u - \mu^2 u - Vu$, $u(0, x) = \phi(x)$, $u_t(0, x) = \psi(x)$ satisfies*

$$\| u(t, \cdot) \|_{W^{k,p}} \le C_{kp} | t |^{1+m(1/p-1/q)} (\| \psi \|_{W^{k,q}} + \| \sqrt{H + \mu^2} \phi \|_{W^{k,q}}).$$

Moreover, if $k \le l - 1$, $\| \sqrt{H + \mu^2} \phi \|_{W^{k,q}}$ may be replaced by $\| \phi \|_{W^{k+1,q}}$.

Another consequence of Theorem 1.1 is on a multiplier theorem for the generalized Fourier transforms. We assume (2) of Assumption 1.1. Then, for any $\xi \in R^m \setminus \{0\}$, there exists a unique solution of $(-\Delta + V(x))\phi_\pm(x, \xi) = | \xi |^2 \phi_\pm(x, \xi)$ satisfying the radiation condition: $\phi_\pm(x, \xi) = e^{ix\cdot\xi} + e^{\pm i|x||\xi|} | x |^{-(m-1)/2} (g(\hat{x}, \xi) + O(| x |^{-1}))$. Define

$$\mathscr{F}_\pm u(\xi) = (2\pi)^{-m/2} \int_{R^m} \overline{\phi_\pm(x, \xi)} u(x) dx.$$

$\mathscr{F}_\pm$ are unitary from $L_c^2(H)$ onto $L^2(R^m)$; $W_\pm = \mathscr{F}_\mp^* \mathscr{F}$; and $\mathscr{F}_\pm$ diagonalize H_c : $\mathscr{F}_\pm H_c \mathscr{F}_\pm^* g(\xi) = | \xi |^2 g(\xi)$. Identity $\mathscr{F}_\pm^* \mathscr{F}_\pm = P_c(H)$ is equivalent to the generalized eigenfunction expansions: $u(x) = (2\pi)^{-m/2} \int_{R^m} \phi_\pm(x, \xi) \mathscr{F}_\pm u(\xi) d\xi$, $u \in L_c^2(H)$.

For a function f on R^m, M_f is the multiplication operator with $f(\xi)$ and $f(D) = \mathscr{F}^* M_f \mathscr{F}$.

Theorem 1.5. *Let V satisfy Assumption 1.1, (2) and let $1 \le p, q \le \infty$. Then for any Borel function f, we have*

$$C^{-1} \| f(D) \|_{B(W^{l,p}, W^{l,q})} \le \| \mathscr{F}_\pm^* M_f \mathscr{F}_\pm \|_{B(W^{l,p}, W^{l,q})} \le C \| f(D) \|_{B(W^{l,p}, W^{l,q})},$$

where the constant C is independent of f.

Combining Theorem 1.5 with known Fourier multiplier theorems, we can give explicit conditions on $f(\xi)$ for $\mathscr{F}_\pm^* M_f \mathscr{F}_\pm$ to be bounded in $W^{l,p}$.

2. Outline of the proof of Theorem 1.1. We prove the case $l = 0$ first. We treat W_+ only. $R(z) = (H - z)^{-1}$ and $R_0(z) = (H_0 - z)^{-1}$ are the resolvents. Assumption 1.1 implies $\| V \|_{L^{(m\pm\varepsilon)/2}} \le CC(V)$, $\varepsilon > 0$. We decompose $V(x) = A(x)B(x)$ with $A, B \in L^{m+\varepsilon} \cap L^{m-\varepsilon}$. Then A and B are super-smooth ([7]) and Kato's theory of smooth operator ([6]) implies that $W_+ f$ can be written as $W_+ f = \sum_{n=0}^N (-1)^n W_n f + L_N f$ for $N = 0, 1 \cdots$. Here W_0 is the identity operator, W_n^*, $n \ge 1$, which we estimate rather than W_n itself, and L_N are written as

$$(2.1) \quad W_n^* f = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} R_0(\lambda + i0) (V R_0(\lambda - i0))^n f d\lambda$$

$$= (-i)^n \int_{[0,\infty)^n} e^{itH_0} V e^{-it_1 H_0} V \cdots V e^{-it_n H_0} f\, dt_1 \cdots dt_n, \quad t = t_1 + \cdots + t_n,$$

$$(2.2) \quad L_N f = \frac{1}{2\pi i} \int_0^\infty (R_0(\lambda - i0)V)^N R(\lambda - i0) \times V\{R_0(\lambda + i0) - R_0(\lambda - i0)\} f d\lambda.$$

Moreover, $\| W_n \|_{B(L^2)} \le (CC(V))^n$ and $\| L_n \|_{B(L^2)} \le (CC(V))^{n+1}$ for $n = 0,1, \cdots$ and, when $C(V)$ is small, $W_+ = \sum_{n=0}^\infty (-1)^n W_n$ converges in the operator norm in $B(L^2)$.

Suppose for the moment $\hat{V} \in C_0^\infty$. Set $K_n(k_1, \ldots, k_n) = i^n (2\pi)^{-nm/2} 2^{-n} \Pi_{j=1}^n \hat{V}(k_j - k_{j-1})$, where $k_0 = 0$. Define for $(t_1, \ldots, t_n) \in R^n$ and $(\omega_1, \ldots, \omega_n) \in \Sigma^n$, Σ is the unit sphere of R^m:

$$\hat{K}_n(t_1, \ldots, t_n, \omega_1, \ldots, \omega_n) = \int_{[0,\infty)^n} e^{-i\Sigma_{j=1}^n t_j s_j /2} (s_1 \cdots s_n)^{m-2} K(s_1\omega_1, \ldots, s_n\omega_n) ds_1 \cdots ds_n.$$

Lemma 2.1. *For* $n = 1,2, \ldots,$ $W_n^* f(x)$ *can be written in the form*

$$(2.3) \quad W_n^* f(x) = \int_{[0,\infty)^{n-1} \times I \times \Sigma^n} \hat{K}_n(t_1, \ldots, t_{n-1}, \tau, \omega_1, \ldots, \omega_n) \times f(\bar{x} + \rho) dt_1 \cdots dt_{n-1} d\tau d\omega_1 \cdots d\omega_n$$

where, $\bar{y} = x - 2(\omega_n \cdot y)\omega_n$ *is the reflection of* y *along the* ω_n *axis,* $\rho = t_1 \bar{\omega}_1 + \cdots + t_{n-1}\bar{\omega}_{n-1} - \tau\omega_n$, *and where* $I = (-\infty, 2\omega_n x - \sigma)$, $\sigma = 2\omega_n(x + t_1\omega_1 + \cdots + t_{n-1}\omega_{n-1})$, *is the range of the integration by the variable* τ.

Note that ρ does not depend on x. Extending the range of integration by τ to the whole line after taking the absolute values of both sides of (2.3), we have

$$| W_n^* f(x) | \le \int_{[0,\infty)^{n-1} \times R \times \Sigma^n} | \hat{K}_n(t_1, \ldots, t_{n-1}, \tau, \omega_1, \ldots, \omega_n) \times f(\bar{x} + \rho) | dt_1 \cdots dt_{n-1} d\tau d\omega_1 \cdots d\omega_n.$$

Noting that $x \to \bar{x}$ is an isometry, we obtain by applying Minkowski's inequality:

Lemma 2.2. *We have* $\| W_n^* f \|_{L^p} \le 2 \| \hat{K}_n \|_{L^1([0,\infty)^n \times \Sigma^n)} \| f \|_{L^p}$, $1 \le p \le \infty$, *for* $n = 1,2, \cdots$.

Let $X = L^{m-1}([0, \infty)^n, L^1(\Sigma^n))$. We have $\| \hat{K}_n \|_X \le C^n \| K_n \|_{L^{m*}(R^{mn})} \le C^n \|\hat{V}\|_{L^{m*}}^n$ by Hölder and Hausdorff-Young inequalities and likewise $\| (\Pi_{j=1}^n \langle t_j \rangle) \hat{K}_n \|_X \le C^n \| \mathcal{F}(\langle x \rangle^2 V) \|_{L^{m*}}^n$. Hence, by the multilinear interpolation inequality and the assumption $\sigma > 2/m_*$, we obtain

$$\| \hat{K}_n \|_{L^1([0,\infty)^n, L^1(\Sigma^n))} \le C^n \| (\prod_{j=1}^n \langle t_j \rangle^{\sigma/2}) \hat{K}_n \|_X \le (C_1 C(V))^n.$$

Combining this with Lemma 2.2, we thus have

Lemma 2.3. *We have* $\| W_n^* f \|_{L^p} \le C_1 (C_2 C(V))^n \| f \|_{L^p}$, $1 \le p \le \infty$.

It is easy to see by the density argument that Lemma 2.3 extends to any V with $\mathcal{F}(\langle x \rangle^\sigma V) \in L^{m*}$ and, if $C(V)$ is small, the series $W_+ = \sum_{n=0}^\infty (-1)^n W_n$ converges in the operator norm in $B(L^p)$. This proves Theorem 1.1 in the case $C(V)$ is small.

Suppose Assumption 1.1, (2) now. $m = 2m' - 1$. Since W_j, $j = 1, \ldots, m$, are bounded in L^p as shown above, we have only to show that L_m is also bounded in L^p. We estimate its integral kernel $L(x, y)$. Setting $N(k) = \{R_0(k^2 - i0)V\}^{m'-1}R(k^2 - i0)\{VR_0(k^2 - i0)\}^{m'-1}$, we rewrite:

(2.4) $$L_m = \frac{1}{\pi i}\int_0^\infty R_0(k^2 - i0)VN(k)V\{R_0(k^2 + i0) - R_0(k^2 - i0)\}kdk.$$

Let $G_\pm(x, k) = \pm(i/4(2\pi)^\nu)|x|^{2-m}(k|x|)^\nu H_\nu^{(j)}(k|x|)$ be the outgoing (incoming) fundamental solutions of $-\Delta - k^2$, where $H_\nu^{(j)}(r)$, $\nu = (m-2)/2$, is the Hankel function of j-th kind and $j = 1$ for $+$ case and $j = 2$ for $-$. The integral kernel of $R_0(k^2 \pm i0)$ is given by $G_{\pm,x,k}(y) = G_\pm(x - y, k)$, and, consequently, that of the integrand of (2.4) is given by $(N(k)V(G_{+,y,k} - G_{-,y,k}), VG_{+,x,k})$. Set $G_{\pm,x,k}(y) = e^{\pm ik|x|}\tilde{G}_{\pm,x,k}(y)$ and define $T_\pm(x, y, k) = (N(k)V\tilde{G}_{\pm,y,k}, V\tilde{G}_{+,x,k})$ and

(2.5) $$L_\pm(x, y) = \frac{1}{i\pi}\int_0^\infty e^{-ik(|x|\mp|y|)}T_\pm(x, y, k)kdk,$$

so that $L(x, y) = L_+(x, y) - L_-(x, y)$. The mapping properties of $R(\lambda \pm i0)$ and $R_0(\lambda \pm i0)$ given in [9] and [3] imply

Lemma 2.4. *Let $j = 0, \ldots, m' + 1$, $\gamma, \gamma' > j + 1/2$ and $0 \le s, s' \le m' - 2$. Then $\langle x\rangle^{-\gamma}N(k)\langle x\rangle^{-\gamma'}$ is a $B(W^{-s',2}, W^{s,2})$-valued C^j-function of k and*

(2.6) $$\|(d/dk)^j\langle x\rangle^{-\gamma}N(k)\langle x\rangle^{-\gamma'}\|_{B(W^{-s',2},W^{s,2})} \le C\langle k\rangle^{-(m-s-s')}.$$

We estimates $L_\pm(x, y)$ by performing integration by parts in (2.5), using Lemma 2.4, the decay property as $|x| \to \infty$ of $\|\langle y\rangle^{-\gamma}(d/dx)^j\tilde{G}_{+,x,k}\|_{L^q(R^m)}$ for suitable γ, j and q and the fact that $e^{\mp ir}r^\nu H_\nu^{(j)}(r)$ are polynomials of degree $m' - 1$ for odd $m = 2m' - 1$. Cancellations of the boundary terms of the integral (2.5) for $L_+(x, y)$ and $L_-(x, y)$ occur and we obtain

Lemma 2.5. *We have* $\sup_{y\in R^m}\int_{R^m}|L(x, y)|dx < \infty$ *and* $\sup_{x\in R^m}\int_{R^m}|L(x, y)|dy < \infty$.

This is a well known criterion for L_m to be bounded in L^p and concludes the proof of Theorem 1.1 for the case $l = 0$.

When $l = 1$, we compute D_jW_nf and D_jL_mf, $j = 1, \ldots, n$. For example, using the first equation of (2.1), $D_jW_nf - W_nD_jf$ can be computed as

$$\sum_{j=0}^{n}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(R_0(\lambda - i0)V)^jR_0(\lambda - i0)(D_jV)(R_0(\lambda - i0)V)^{n-j-1}$$
$$R_0(\lambda + i0)fd\lambda.$$

This is a sum of terms which have exactly the same form as the adjoint of the second of (2.1) except that one of V is replaced by D_jV. Thus, the argument which leads to Lemma 2.3 above implies

$$\|D_jW_nf\|_{L^p} \le (n+1)(C_1C(V))^{n-1}(C(V) + C(D_jV))\|f\|_{W^{1,p}},\quad 1 \le p \le \infty.$$

This shows that the series $\sum_{n=1}^\infty W_n$ converges in the operator norm in $B(W^{1,p})$ for the same value of $C(V)$ for which it converges in $B(L^p)$. Thus W_+ is bounded in $W^{1,p}$. The argument for D_jL_mf is similar and Theorem 1.1 holds for $l = 1$. For general $l \ge 2$, we repeat this argument. The detail will appear elsewhere.

Acknowledgement. The author expresses his hearty thanks to Professor Tosio Kato for teaching him a time dependent proof of (2.3) and Lemma 2.2 based on the second equation of (2.1). The original proof was based on the first of (2.1).

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