2. On Foliations on Complex Spaces. II

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§0. Introduction. We defined coherent foliations on reduced complex spaces in the previous papers [5,6]. In this paper, we discuss foliations of dimension one or of codimension one on reduced complex spaces, especially on locally irreducible complex spaces and show that our definition is an extension of the definition by Gómez-Mont [2] of foliations by curves, i.e. the foliation whose leaves are of complex dimension one. Details of proofs are described in [5].

§1. Foliations of dimension one or of codimension one. Let $(X, \mathcal{O}_X)$ be a reduced complex space. We use the following notations:

$\mathcal{M}_X$: the sheaf of germs of meromorphic functions on X

Ω_X: the sheaf of germs of holomorphic 1-forms on X

Θ_X: the sheaf of germs of holomorphic vector fields on X

sp$X$: the underlying topological space of the complex space X.

For a coherent $\mathcal{O}_X$-module $\mathcal{S}$, we set

$$\mathrm{Sing}\mathcal{S} := \{x \in X \mid \mathcal{S}_x \text{ is not } \mathcal{O}_{X,x}\text{-free}\}.$$

For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, we use the notation:

$$S(\mathcal{T}) := \mathrm{Sing}\mathcal{S} \cup \mathrm{Sing}(\mathcal{S}/\mathcal{T}).$$

$S(\mathcal{T})$ is an analytic set in X satisfying $S(\mathcal{T}) \supset \mathrm{Sing}\mathcal{T}$. On $X - S(\mathcal{T})$, $\mathcal{T}$ is locally a direct summand of $\mathcal{S}$.

Our definition of foliations on complex spaces is as follows:

Definition 1.0. Definition a) (by 1-forms).

a.0) A *coherent foliation* on X is a coherent $\mathcal{O}_X$-submodule F of Ω_X satisfying

(1.1) $$dF_x \subset F_x \wedge \Omega_{X,x}$$

at any $x \in X - S(F)$. This condition is called the *integrability condition.* We call $S(F)$ the *singular locus* of the foliation F.

a.1) A coherent foliation $F \subset \Omega_X$ is said to be *reduced* if, for any open subspace $U \subset X$, $\xi \in \Gamma(U, \Omega_X)$ and $\xi|_{U-S(F)} \in \Gamma(U - S(F), F)$ imply $\xi \in \Gamma(U, F)$.

Definition b) (by vector fields).

b.0) A *coherent foliation* on X is a coherent $\mathcal{O}_X$-submodule E of Θ_X satisfying

(1.2) $$[E_x, E_x] \subset E_x$$

at any $x \in X - (S(E) \cup \mathrm{Sing}X)$. This condition is called the *intergrability condition.*

We call $S(E) \cup \mathrm{Sing}X$ the *singular locus* of the foliation E.

b.1) A coherent foliation $E \subset \Theta_X$ is said to be *reduced* if, for any open subspace $U \subset X$, $v \in \Gamma(U, \Theta_X)$ and $v|_{U-(S(E)\cup \mathrm{Sing}X)} \in \Gamma(U - (S(E) \cup \mathrm{Sing}X), E)$ imply $v \in \Gamma(U, E)$.

These two definitions are related with each other. Namely,

Definition 1.3. 0) For a coherent foliation $F \subset \Omega_X$, we define a coherent foliation $F^a \subset \Theta_X$ by

$$F_x^a := \{v \in \Theta_{X,x} \mid \langle v, \xi \rangle = 0 \text{ for all } \xi \in F_x\}.$$

1) For a coherent foliation $E \subset \Theta_X$, we define a coherent foliation $E^\perp \subset \Omega_X$ by

$$E_x^\perp := \{\xi \in \Omega_{X,x} \mid \langle v, \xi \rangle = 0 \text{ for all } v \in E_x\}.$$

Theorem 1.4. 0) *For a coherent foliation $F \subset \Omega_X$, the $\mathcal{O}_X$-submodules F^a of Θ_X and $F^{a\perp} = (F^a)^\perp$ of Ω_X are reduced coherent foliations.*
1) *For a coherent foliation $E \subset \Theta_X$, the $\mathcal{O}_X$-submodules $E^\perp$ of Ω_X and $E^{\perp a} = (E^\perp)^a$ of Θ_X are reduced coherent foliations.*
2) *These correspondences*

$$F \subset \Omega_X \rightarrow F^a \subset \Theta_X \text{ and } E \subset \Theta_X \rightarrow E^\perp \subset \Omega_X$$

restricted to reduced coherent foliations are inverse of each other.

Now let X be a reduced irreducible complex space. For any coherent $\mathcal{O}_X$-module $\mathcal{S}$, $\mathrm{Sing}\mathcal{S}$ is a thin analytic set in X and $X - \mathrm{Sing}\mathcal{S}$ is connected. Thus the rank of locally free sheaf $\mathcal{S}|_{X-\mathrm{Sing}\mathcal{S}}$ on $X - \mathrm{Sing}\mathcal{S}$ is well defined, which we call the *rank* of the coherent sheaf $\mathcal{S}$.

Definition 1.5. 0) A coherent foliation $E \subset \Theta_X$ is called *of dimension p* if the coherent $\mathcal{O}_X$-module E is of rank p.
1) A coherent foliation $F \subset \Omega_X$ is called *of codimension q* if the coherent $\mathcal{O}_X$-module F is of rank q.
2) A coherent foliation $E \subset \Theta_X$ is said *of codimension q* if $E^\perp$ is of codimension q.
3) A coherent foliation $F \subset \Omega_X$ is said *of dimension p* if F^a is of dimension p.

Remarks. A coherent foliation $E \subset \Theta_X$ of dimension p defines a non-singular foliation of dimension p on the (connected) complex manifold $X - (S(E) \cup \mathrm{Sing}X)$. A coherent foliation $F \subset \Omega_X$ of codimension q defines a non-singular foliation of codimension q on the (connected) complex manifold $X - S(F)$.

We recall the following definitions.

Definition 1.6. 0) A complex space X is said to be *of pure dimension n at $x \in X$* if every prime component of X at x is of dimension n at x.
1) X is called *of pure dimension n* if X is of pure dimension n at every $x \in X$.
2) X is *non-singular in codimension one* if the singular locus $\mathrm{Sing}X$ of X is of codimension strictly greater than one in X.

Remark. On a locally irreducible complex space X, a coherent sheaf $\mathcal{S}$ is of rank r if the restriction of $\mathcal{S}$ to each component of X is coherent of rank r.

Theorem 1.7. 0) *On a complex manifold M, every reduced coherent foliation $E \subset \Theta_M$ of dimension one is an invertible subsheaf of Θ_M.*
1) *Let $(X, \mathcal{O}_X)$ be a locally irreducible complex space and A an analytic set in X thin of order two and containing* $\mathrm{Sing}X$. *For any coherent foliation $E_0 \subset \Theta_{X-A}$ on $X - A$ of dimension one, there exists a uniquely determined reduced foliation*

$E \subset \Theta_X$ on X of dimension one satisfying $E|_{X-(A\cup S(E_0))} = E_0|_{X-(A\cup S(E_0))}$.
To prove this theorem, we need some preliminaries.

§2. Another characterization of the coherent subsheaf $\mathcal{T}_F$. Let $(X, \mathcal{O}_X)$ be a reduced complex space, $\mathcal{S}$ a coherent $\mathcal{O}_X$-module and $\mathcal{T}$ a coherent $\mathcal{O}_X$-submodule of $\mathcal{S}$. Recall the notations in [6] §2.

Definition 2.0. Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{S}^*$ the dual $\mathcal{O}_X$-module of $\mathcal{S}$.
0) For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, an $\mathcal{O}_X$-submodule $\mathcal{T}_F$ of $\mathcal{S}$ is defined by a complete presheaf

$$\mathcal{T}_F(U) := \{\xi \in \mathcal{S}(U) \mid \xi|_{U-S(\mathcal{S})} \in \mathcal{T}(U - S(\mathcal{S}))\} \text{ for } U \subset X.$$

1) For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, $\mathcal{T}^a$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{S}^*$ whose stalk on $x \in X$ is

$$\mathcal{T}^a_x := \{v \in \mathcal{S}^*_x \mid \langle v, \xi\rangle = 0 \text{ for all } \xi \in \mathcal{T}_x\}.$$

2) For a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$, we define a coherent $\mathcal{O}_X$-submodule $\mathcal{R}^\perp$ of $\mathcal{S}$ by

$$\mathcal{R}^\perp_x := \{\xi \in \mathcal{S}_x \mid \langle v, \xi\rangle = 0 \text{ for all } v \in \mathcal{R}_x\}.$$

Proposition 2.1. *Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{S}^*$ the dual $\mathcal{O}_X$-module of $\mathcal{S}$.*
0) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, $\mathcal{T}_F$ satisfies*
a) $\mathcal{T} \subset \mathcal{T}_F$.
b) $\mathcal{T}|_{X-S(\mathcal{T})} = \mathcal{T}_F|_{X-S(\mathcal{T})}$.
c) *If a section $\xi \in \Gamma(U, \mathcal{S})$ on an open subset $U \subset X$ satisfies, for some thin analytic set A in U, $\xi|_{U-A} \in \Gamma(U-A, \mathcal{T})$, then $\xi \in \Gamma(U, \mathcal{T}_F)$.*
1) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$,*

$$\mathcal{T}_F = \mathcal{T}^{a\perp} (:= (\mathcal{T}^a)^\perp).$$

Thus $\mathcal{T}_F$ is coherent.
2) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$,*

$$\mathcal{R}_F = \mathcal{R}^{\perp a} (:= (\mathcal{R}^\perp)^a).$$

Here, for a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of a coherent $\mathcal{O}_X$-module $\mathcal{S}$, we define another coherent submodule $\mathcal{T}_{\mathcal{M}} \subset \mathcal{S}$, which turns out to be coincide with $\mathcal{T}_F$ if the complex space X is locally irreducible. Note that the functor $\otimes_{\mathcal{O}X}\mathcal{M}_X$ is exact in the category of $\mathcal{O}_X$-modules, i.e. for any exact sequence of (not necessalily coherent) $\mathcal{O}_X$-modules

$$0 \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to 0,$$

the induced sequence

$$0 \to \mathcal{S}_0 \otimes_{\mathcal{O}X} \mathcal{M}_X \to \mathcal{S}_1 \otimes_{\mathcal{O}X} \mathcal{M}_X \to \mathcal{S}_2 \otimes_{\mathcal{O}X} \mathcal{M}_X \to 0$$

is also exact.

For any $\mathcal{O}_X$-submodule $\mathcal{S}$, let

$$\mu_{\mathcal{S}} : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{O}X} \mathcal{M}_X$$

be the canonical $\mathcal{O}_X$-morphism.

Now we define the $\mathcal{O}_X$-submodule $\mathcal{T}_{\mathcal{M}}$. For an $\mathcal{O}_X$-module $\mathcal{S}$ and an $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, we have the following commutative diagram, whose rows are exact:

$$\begin{array}{ccccccccc} 0 \to & \mathcal{T} & \to & \mathcal{S} & \to & \mathcal{Q} & \to 0 \\ & \mu_{\mathcal{T}} \downarrow & & \mu_{\mathcal{S}} \downarrow & & \mu_{\mathcal{Q}} \downarrow & \\ 0 \to & \mathcal{T} \otimes_{\mathcal{O}X} \mathcal{M}_X & \to & \mathcal{S} \otimes_{\mathcal{O}X} \mathcal{M}_X & \to & \mathcal{Q} \otimes_{\mathcal{O}X} \mathcal{M}_X & \to 0 \end{array},$$

where $\mathcal{Q} = \mathcal{S}/\mathcal{T}$ is the quotient $\mathcal{O}_X$-module. By this diagram, we consider $\mathcal{T} \otimes_{\mathcal{O}X} \mathcal{M}_X$ as an $\mathcal{O}_X$-submodule of $\mathcal{S} \otimes_{\mathcal{O}X} \mathcal{M}_X$.

Definition 2.2. For an $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, $\mathcal{T}_{\mathcal{M}}$ is the $\mathcal{O}_X$-submodule of $\mathcal{S}$ defined by

$$\mathcal{T}_{\mathcal{M}} := \mu_{\mathcal{S}}^{-1}(\mathcal{T} \otimes_{\mathcal{O}X} \mathcal{M}_X).$$

Lemma 2.3. *As $\mathcal{O}_X$-submodules of $\mathcal{S}$, the following holds:*

0) $(\mathcal{T}_{\mathcal{M}})_{\mathcal{M}} = \mathcal{T}_{\mathcal{M}}$.

1) $\mathcal{T} \subset \mathcal{T}_{\mathcal{M}}$.

2) $\mathcal{T}|_{X-S(\mathcal{T})} = \mathcal{T}_{\mathcal{M}}|_{X-S(\mathcal{T})}$.

3) $\mathcal{T} \subset \mathcal{T}_{\mathcal{M}} \subset \mathcal{T}^{a\perp}$.

On locally irreducible complex spaces, a stronger assertion holds:

Theorem 2.4. *Let $(X, \mathcal{O}_X)$ be a locally irreducible complex space, $\mathcal{S}$ a coherent $\mathcal{O}_X$-module and $\mathcal{T}$ a coherent $\mathcal{O}_X$-submodule of $\mathcal{S}$. Let $\mathcal{T}_F$, $\mathcal{T}^{a\perp}$ and $\mathcal{T}_{\mathcal{M}}$ be the $\mathcal{O}_X$-submodules of $\mathcal{S}$ defined in* [6] §2 *and Definition* 2.2, *respectively. Then*

$$\mathcal{T}_F = \mathcal{T}^{a\perp} = \mathcal{T}_{\mathcal{M}}.$$

The proof is given by algebraic observations in appendices of [5]. Theorem 1.7 follows easily from Theorem 2.4.

§3. Relation with Gómez-Mont's definition. In [2,3], Gómez-Mont defined foliations by curves on complex spaces as follows:

Definition 3.0 (Gómez-Mont). 0) A foliation E by curves on a complex manifold M is defined in the following two ways, which are equivalent to each other.

a) An invertible $\mathcal{O}_M$-submodule of Θ_M.

b) A (natural equivalence class of a) morphism $\Phi : L \to T$ of holomorphic vector bundle on M not identically zero on each component of M, where L is a holomorphic line bundle on M and $T := TM$ is the holomorphic tangent bundle of M. Two such morphisms $\Phi_1 : L_1 \to T$ and $\Phi_2 : L_2 \to T$ are equivalent if there are an isomorphism $\psi : L_1 \to L_2$ and a never vanishing holomorphic function $\lambda \in \mathcal{O}_M(M)$ such that the following diagram commutes:

$$\begin{array}{cccc} & L_1 & \xrightarrow{\Phi_1} & T \\ \psi & \downarrow & & \downarrow & \lambda\cdot. \\ & L_2 & \xrightarrow[\Phi_2]{} & T \end{array}$$

1) Let $(X, \mathcal{O}_X)$ be a reduced complex space of pure dimension and non-singular in codimension one. A foliation F by curves on X is

c) A (natural equivalence class of a) pair (F_A, A), where A is an analytic set in X thin of order two and containing Sing X and F_A is a non-singular foliation on the complex manifold $X - A$. Two such pairs (F_A, A) and (F_B, B) are equivalent if

$$F_A|_{X-(A\cup B)} = F_B|_{X-(A\cup B)}.$$

Remarks. 0) This definition of foliations on manifolds may seem rather strict. In the case of dimension one, Baum-Bott [1] and Suwa [7,8] defined them to be *coherent subsheaves of rank one* of Θ_M, not neccesarily invertible. Theorem 1.7 tells us that these definitions are essentially equivalent to one another.

1) It follows from Theorem 1.7 that, on locally irreducible complex spaces which are of pure dimension and non-singular in codimension one, the above definition is equivalent to our definition of coherent foliations of dimension one. Considering an embedding into a domain in some complex number space, Gómes-Mont proved this in 1) of Theorem 5 of [2] (pp. 131-132), which is written in our language as follows:

Proposition 3.1 (Gómez-Mont). *Let $E \subset \Theta_X$ be a coherent foliation of dimension one on a normal complex space X. Assume the germ of the zero locus of $v \in E_x$ at $x \in X$ is of codimension strictly greater than one. Then E is reduced at x, invertible at x and written $E_x = \mathcal{O}_{X,x}v$.*

Our method is applicable to the case of foliations of codimension one as we see below.

Definition 3.2 (Suwa [7] p. 184 (1.10) Definition). A foliation $F \subset \Omega_M$ on a complex manifold M is *of complete intersection type* if F is a locally free $\mathcal{O}_M$-module.

As Theorem 1.7, we have

Proposition 3.3. 0) *On a complex manifold M, every reduced coherent foliation $F \subset \Omega_M$ of codimension one is of complete intersection type.*

1) *Let $(X, \mathcal{O}_X)$ be a locally irreducible complex space and A an analytic set in X thin of order two and containing* Sing$X$. *For any coherent foliation $F_0 \subset \Omega_{X-A}$ on $X - A$ of codimension one, there exists a uniquely determined reduced foliation $F \subset \Omega_X$ on X of codimension one satisfying $F|_{X-(A\cup S(F_0))} = F_0|_{X-(A\cup S(F_0))}$.*

References

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