91. *On the Region Free from the Poles of the Resolvent for the Elastic Wave Equation with the Neumann Boundary Condition*

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1. Introduction. Let Ω be an exterior domain in $\boldsymbol{R}^n (n \geq 3)$ with smooth and compact boundary Γ. We consider the isotropic elastic wave equation with the Neumann boundary condition

$$\text{(N)} \qquad \begin{cases} (A(\partial_x) - \partial_t^2)u(t, x) = 0 & \text{in } \boldsymbol{R} \times \Omega, \\ N(\partial_x)u(t, x) = 0 & \text{on } \boldsymbol{R} \times \Gamma, \\ u(0, x) = f_0(x),\ \partial_t u(0, x) = f_1(x) & \text{on } \Omega, \end{cases}$$

where $u(t, x) = {}^t(u_1(t, x), \cdots, u_n(t, x))$ is the displacement vector. Using the stress tensor $\sigma_{ij}(u) = \lambda(\operatorname{div} u)\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and the unit outer normal vector $\nu(x) = {}^t(\nu_1(x), \nu_2(x), \cdots, \nu_n(x))$ to Ω at $x \in \Gamma$, we can give the operator $A(\partial_x)$ and the boundary operator $N(\partial_x)$ by $(A(\partial_x)u)_i = \sum_{j=1}^n \partial_{x_j}(\sigma_{ij}(u))$, $(N(\partial_x)u)_i = \sum_{j=1}^n \nu_j(x)\sigma_{ij}(u)\,|_\Gamma (i = 1, 2, \cdots, n)$. Note that $A(\partial_x)$ can also be written as $A(\partial_x)u = \mu\Delta u + (\lambda + \mu)\operatorname{grad}(\operatorname{div} u)$.

We assume that the Lamé constants λ and μ are independent of the variables t and x and satisfy

$$\lambda + \frac{2}{n}\mu > 0 \text{ and } \mu > 0.$$

We define the outgoing resolvent $R(z)$ of the problem (N) as the solution operator of the reduced elastic wave equation

$$\begin{cases} (A(\partial_x) + z^2)v(x\,; z) = f(x) & \text{in } \Omega, \\ N(\partial_x)v(x\,; z) = 0 & \text{on } \Gamma, \\ v(x\,; z) \text{ is outgoing,} \end{cases}$$

where the word "outgoing" means that $v(x\,; z)$ is the $L^2(\Omega)$-solution if $\operatorname{Im} z < 0$ and the analytic continuation of the $L^2(\Omega)$-solution in the region $\operatorname{Im} z < 0$ if $\operatorname{Im} z \geq 0$. Note that for any $a > 0$ with $\Gamma \subset B_a = \{x \in \boldsymbol{R}^n \,|\, |x| < a\}$, $R(z)$ is a $B(L_a^2(\Omega), H^2(\Omega \cap B_a))$-valued meromorphic function in $\tilde{C}$, and a $B(L_a^2(\Omega), H^2(\Omega \cap B_a))$-valued holomorphic function in $\operatorname{Im} z \leq 0$, $z \neq 0$, where $L_a^2(\Omega) = \{f \in L^2(\Omega) \mid \operatorname{supp} f \subset \Omega \cap B_a\}$, and $\tilde{C} = C$ if n is odd, $\tilde{C} = \left\{z \in C \setminus \{0\} \mid -\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi\right\}$ if n is even (cf. Iwashita and Shibata [3]).

The purpose of this note is to give some information about the location of the poles of the outgoing resolvent of the problem (N). For the problem (N), it is well known that there exists the Rayleigh surface wave propagating along the boundary (see Achenbach [1] or Taylor [6]). This fact suggests that the poles of $R(z)$ approaching to the real axis come out. Furthermore, we can expect that the poles of the Rayleigh surface wave stated above do not appear away from the real axis. In this note, we succeed in the justification of the latter expectation although we can not prove the former one.

In the case of the scalar-valued wave equation with the Dirichlet or the Neumann boundary condition or the elastic wave equation with the Dirichlet boundary condition, if all of the waves go away from the boundary as the time tends to infinity, then the resolvent defined by the same way as that of the problem (N) is holomorphic in the region $|\operatorname{Im} z| < \alpha \log |\operatorname{Re} z| - \beta$ with fixed constants α and $\beta > 0$ (see e. g. Iwashita and Shibata [3], Vainberg [7]). Hence for the problem (N), we may prove that there are constants α and $\beta > 0$ such that all of the poles of $R(z)$ in the region $|\operatorname{Im} z| < \alpha \log |\operatorname{Re} z| - \beta$ are caused by the Rayleigh surface wave and those poles do not spread out from a neighborhood of the real axis if any wave except the Rayleigh surface wave does not remain near the boundary. About this expectation, we have the following result about the region free from the poles.

Theorem 1. *If the boundary Γ is strictly convex, then for any integer $j \geq 0$, there is a constant $c_j > 0$ such that the outgoing resolvent $R(z)$ is holomorphic in the region*

$$D_j = \{z \in U_{\alpha\beta} \mid \operatorname{Im} z \geq c_j |\operatorname{Re} z|^{-j}\},$$

where $U_{\alpha\beta} = \{z \in \tilde{C} \,||\operatorname{Im} z| < \alpha \log |\operatorname{Re} z| - \beta\}$ and the constants $\alpha, \beta > 0$ depend only upon Γ, λ and μ.

Remark. Recently Stefanov and Vodev [5] show the very precise result about the distribution of the poles of $R(z)$ when the boundary Γ is the unit sphere in $\boldsymbol{R}^3$. In [5], they show that the poles of $R(z)$ are equal to the zeros of the holomorphic functions which are given by using the spherical Hankel functions of first order. Hence, in their approach, it is indispensable to assume that the boundary is the sphere since they have to represent the solution by using special functions.

We can also get the estimate of $R(z)$ in D_j.

Theorem 2. *If the boundary Γ is strictly convex, then for any $a > 0$ with $\Gamma \subset B_a$, there are constants $C_a > 0$ and $T_a > 0$ such that for any integer $j \geq 0$ we have*

$$\| R(z) f \|_{H^{2-l}(\Omega \cap B_a)} \leq C_a |\operatorname{Im} z|^{-1} |z|^{9-l} e^{T_a |\operatorname{Im} z|} \| f \|_{L^2(\Omega)}$$

$$\text{for any } f \in L^2_a(\Omega),\ l = 0, 1, 2,\ z \in D_j.$$

Theorem 2 is useful to get the asymptotic behaviour of the solution of the problem (N) as the time goes to infinity. If any solution of (N) with compact supported datum decays exponentially, then Theorem 2 implies that the same arguments as in Vainberg [7] is valid for the problem (N). Thus we can show that the problem (N) has the uniform local-energy decay property of strong type in the sense of Definition 0.1 in Kawashita [4]. But this conclusion contradicts Theorem 0.2 in [4]. Hence, we can get the following result.

Corollary 1. *If the boundary Γ is strictly convex, then for any $a>0$ with $\Gamma\subset B_a$, there is a solution $u(t, x)$ of the problem* (N) *with* $\operatorname{supp} f_0\cup\operatorname{supp} f_1\subset\bar{\Omega}\cap B_a$ *satisfying that the local energy of $u(t, x)$ does not decay exponentially.*

Note that Ikehata and Nakamura [2] have already shown the nondecaying property of the local energy by using special functions when the boundary Γ is the unit sphere in $\boldsymbol{R}^3$.

2. On proof of the theorems. We begin to introduce the Neumann operator which plays an important role in proof of Theorems 1 and 2.

We denote by $U^+(z)g$ (resp. $U^-(z)g$) the outgoing (resp. incoming) solution of the reduced elastic wave equation with an inhomogeneous Dirichlet datum $g\in H^{3/2}(\Gamma)$. Then we define the Neumann operator $T^\pm(z)$ as $T^\pm(z)g=N(\partial_x)U^\pm(z)g$. It is well known that $T^\pm(z)$ is a $B(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$-valued holomorphic function in $U_{\alpha\beta}$ for some $\alpha, \beta>0$ if the boundary Γ is strictly convex (cf. Iwashita and Shibata [3]). Note that the pole z_0 of $R(z)$ contained in $U_{\alpha\beta}$ is characterized as the point satisfying Ker $T^+(z_0)\neq\{0\}$, which is equivalent to the fact that z_0 is a pole of $(T^+(z))^{-1}$ as a $B(H^{1/2}(\Gamma), H^{3/2}(\Gamma))$-valued function. Hence, in our approach deducing Theorems 1 and 2, it is very important to get the estimates of $T^\pm(z)$ from below.

Proposition 1. *If the boundary Γ is strictly convex, then there are constants $C=C(\Gamma, \lambda, \mu)>0$, $T=T(\Gamma, \lambda, \mu)>0$ such that*

$$\|g\|_{H^{-1/2}(\Gamma)}\leq C|\operatorname{Im} z|^{-1}|z|^6 e^{T|\operatorname{Im} z|}\|T^\pm(z)g\|_{H^{-3/2}(\Gamma)} + C_j|\operatorname{Im} z|^{-1}|z|^{6-j}e^{T|\operatorname{Im} z|}\|g\|_{H^{-1/2}(\Gamma)}$$

for any $g\in C^\infty(\Gamma)$, $z\in U_{\alpha\beta}\setminus\boldsymbol{R}$ and integer $j\geq 0$,

where $C_j>0$ is independent of g and z.

When the boundary Γ is strictly convex, any singularity of the solution of (N) except singularity corresponding to the Rayleigh surface wave does not remain near the boundary (cf. Taylor [6] and Yamamoto [8]). Proposition 1 is based on the above fact.

Using Proposition 1 we have $(T^\pm(z))^{-1}$ is holomorphic in D_j stated in Theorem 1 for any integer $j\geq 0$. Furthermore, we get the estimate of $(T^\pm(z))^{-1}$, if we combine Proposition 1 with the fact that

$$(T^+(z)g, h)_{L^2(\Gamma)}=(g, T^-(\bar{z})h)_{L^2(\Gamma)}$$

for any $g, h\in H^{3/2}(\Gamma)$ and $z\in U_{\alpha\beta}$.

These estimates imply Theorem 2. Thus in our approach, the estimates of $T^\pm(z)$ in Proposition 1 are crucial.

In this note, we only show an outline. The detail will be given in a forthcoming paper.

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