88. *Euler Characteristics of Groups and Orbit Spaces of Free G-complexes*

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Abstract: Let Γ be a group of finite homological type, X a finite dimensional, free Γ-complex such that $H_*(X, \mathbf{Z})$ is finitely generated. We proved that $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, and $\chi(X/\Gamma) = \chi(\Gamma)\cdot\chi(X)$, where $\chi(\Gamma)$ is the Euler characteristic of a group Γ.

Key words: Topological transformation groups; Euler characteristics of groups.

Let Γ be a discrete group. By a *Γ-complex* we will mean a CW-complex X together with an action of Γ on X which permutes the cells. A Γ-complex X is said to be *free* if the action of Γ freely permutes the cell of X. This note attempts to provide a formula concerning the Euler characteristics of a finite dimensional, free Γ-complex X and of the orbit space X/Γ. Such a formula is well-known for finite groups [3, p. 245]:

Theorem 1. *Let G be a finite group, X a finite dimensional free G-complex such that $H_*(X, \mathbf{Z})$ is finitely generated. Then $H_*(X/G, \mathbf{Z})$ is finitely generated, and*

$$\chi(X) = |G| \cdot \chi(X/G).$$

On the other hand, let Γ be a fundamental group of a finite aspherical complex $B\Gamma$. Then we see that $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, and $\chi(X/\Gamma) = \chi(B\Gamma)\cdot\chi(X)$. This follows from the product formula of Euler characteristics for fibrations applied to the Borel construction $X \to E\Gamma\times_\Gamma X \to B\Gamma$, where $E\Gamma$ is the universal cover of $B\Gamma$, and from the fact that $H_*(E\Gamma\times_\Gamma X, \mathbf{Z}) \cong H_*(X/\Gamma, \mathbf{Z})$. We will unify and extend these formulae by means of the Euler characteristics of groups.

The Euler characteristics of abstract groups has been studied by a number of authors under different conditions. We employ the one developed in Brown's book [3]. Recall that Γ is a *group of finite homological type* if (i) Γ is a group of finite virtual cohomological dimension (written $\mathrm{vcd}\,\Gamma < \infty$) and (ii) for any $\mathbf{Z}\Gamma$-module M which is finitely generated as a $\mathbf{Z}$-module, $H_n(\Gamma, M)$ is finitely generated for all n. A group Γ is of finite homological type if and only if a subgroup of finite index is.

Given a group Γ of finite homological type, the *Euler characteristic* $\chi(\Gamma)$ of Γ is defined. Namely, when Γ is torsion-free, set

$$\chi(\Gamma) = \sum_i (-1)^i \operatorname{rank}_{\mathbf{Z}} H_i(\Gamma, \mathbf{Z}).$$

When Γ has torsion, set

$$\chi(\Gamma)=\frac{1}{(\Gamma:\Gamma')}\cdot\chi(\Gamma')\in\mathbf{Q},$$

where Γ' is a torsion-free subgroup of finite index, and $(\Gamma:\Gamma')$ is the index of Γ' in Γ. $\chi(\Gamma)$ is indepedent of the choice of Γ'. For more details of groups of finite homological type and Euler characteristics of them, the reader will refer Chapter IX of [3] as well as [1] and [2]. Now we will state our result:

Theorem 2. *Let Γ be a group of finite homological type, X a finite dimensional free Γ-complex such that $H_*(X, \mathbf{Z})$ is finitely generated. Then $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, and*

$$\chi(X/\Gamma)=\chi(\Gamma)\cdot\chi(X).$$

The proof will be done by the use of the Cartan-Leray spectral sequence and Theorem 1. Observe that Theorem 1 is a particular case of Theorem 2. For if G is a finite group, then $\chi(G)=1/|G|$. Or if Γ is a fundamental group of a finite aspherical complex $B\Gamma$, then $\chi(B\Gamma)=\chi(\Gamma)$. Hence Theorem 2 unifies those two cases mentioned above. And Theorem 2 is new if Γ is an infinite group of finite homological type with torsion. Arithmetic groups provide a large number of examples of such groups (cf. [4]).

Proof of Theorem 2. Assume first that Γ is torsion-free. Consider the Cartan-Leray spectral sequence

$$E^2_{pq}=H_p(\Gamma, H_q(X, \mathbf{Z}))\Rightarrow H_{p+q}(X/\Gamma, \mathbf{Z}).$$

For each p and q, E^2_{pq}-term is finitely generated, since $H_q(X, \mathbf{Z})$ is finitely generated and Γ is of finite homological type. And $E^2_{pq}=0$ whenever $p+q>\dim X+\operatorname{cd}\Gamma$, where $\operatorname{cd}\Gamma$ denotes the cohomological dimension of Γ. Hence $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated. When Γ has torsion, choose a torsion-free normal subgroup Γ' of finite index. Then X/Γ' is a finite dimensional, free Γ/Γ'-complex, and the orbit space $(X/\Gamma')/(\Gamma/\Gamma')$ is homeomorphic to X/Γ. Now Theorem 1 proves that $H_*(X/\Gamma, \mathbf{Z})$ is finitely generated, since Γ/Γ' is a finite group and $H_*(X/\Gamma', \mathbf{Z})$ is finitely generated. Before showing the rest of Theorem 2, we will prove a lemma.

Lemma. *Under the hypothesis of Theorem 2, there exists a torsion-free, normal subgroup Γ_0 of finite index such that Γ_0 acts on $H_*(X, \mathbf{Z}/2)$ trivially.*

Proof. Choose a torsion-free subgroup Γ' of finite index. The action of Γ' on $H_*(X, \mathbf{Z}/2)$ yields a group homomorphism $\phi:\Gamma'\to\operatorname{Aut}(H_*(X, \mathbf{Z}/2))$, where $\operatorname{Aut}(-)$ is a group of automorphisms of a graded $\mathbf{Z}/2$-vector space. Obviously a subgroup $\ker\phi$ is torsion-free and acts on $H_*(X, \mathbf{Z}/2)$ trivially. And $\ker\phi$ is of finite index in Γ' since $\operatorname{Aut}(H_*(X, \mathbf{Z}/2))$ is finite. In general, $\ker\phi$ is not normal in Γ. However, since $(\Gamma:\ker\phi)<\infty$, there is a subgroup Γ_0 of finite index in $\ker\phi$ such that Γ_0 is normal in Γ, which is a required Γ_0.

Now suppose Γ_0 is a subgroup of Γ as in Lemma. Consider the Cartan-Leray spectral sequence with coefficients in $\mathbf{Z}/2$:

$$E^2_{pq}=H_p(\Gamma_0, H_q(X, \mathbf{Z}/2))\Rightarrow H_{p+q}(X/\Gamma_0, \mathbf{Z}/2).$$

Since Γ_0 acts on $H_*(X, \mathbf{Z}/2)$ trivially, each E^2_{pq}-term splits into a tensor product as $E^2_{pq}\cong H_p(\Gamma_0, \mathbf{Z}/2)\otimes_{\mathbf{Z}/2}H_q(X, \mathbf{Z}/2)$. Using this isomorphism to calculate $\chi(X/\Gamma_0)$, we obtain $\chi(X/\Gamma_0)=\chi(\Gamma_0)\cdot\chi(X)$. On the other hand, applying Theorem 1 to a finite dimensional, free Γ/Γ_0-complex X/Γ_0, we obtain

$$\chi(X/\Gamma_0)=|\Gamma/\Gamma_0|\cdot\chi((X/\Gamma_0)/(\Gamma/\Gamma_0))=(\Gamma:\Gamma_0)\cdot\chi(X/\Gamma).$$

Combining these equalities together with the definition of $\chi(\Gamma)$, the proof is done.

Remark. In general, $\chi(\Gamma)$ is not an integer, however, $\chi(\Gamma)\cdot\chi(X)$ is. Under the assumption of Theorem 2, it follows from Theorem 1 that $|G|$ divides $\chi(X)$ for any finite subgroup G of Γ. Consequently, the least common multiple m of the orders of finite subgroups of Γ must divide $\chi(X)$. But $m\cdot\chi(\Gamma)$ is an integer [3, p. 257].

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