87. Remark on Kohnen-Zagier's Paper Concerning Fourier Coefficients of Modular Forms of Half Intergral Weight

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Introduction. In [5], Shimura has established a correspondence Ψ between the space of modular forms of half integral weight and the space of those of integral weight. Using the methods and languages of representation theory of adeles of metaplectic groups, Waldspurger [8] and [9] showed that the square of Fourier coefficients $a(n)$ for a square free integer n of the modular form $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ of half integral weight is essentially proportional to the special value of the zeta function at a certain integer attached to the modular form F if f corresponds to F by Ψ and f is an eigen-function of Hecke operators.

On the other hand, Kohnen-Zagier [1] and [2] determined explicitly the proportion of the square of $a(n)$ for a square free integer n of $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ which is an eigen-function of Hecke operators and belongs to the Kohnen's subspace $S^{+}_{(2k+1)/2}(4N) = \{f(z) = \sum_{(-1)^k n \equiv 0,1(4)} a(n)e[nz] \in S_{(2k+1)/2}(4N)\}$ of $S_{(2k+1)/2}(4N)$ by the special value of the zeta function associated with the modular form $F = \Psi(f)$ of integral weight. Kohnen-Zagier [1] (resp. Kohnen [2]) treated the case where $N = 1$ (resp. N is an odd square free integer). The purpose of this note is to derive an analogy of [1] and [2] in the case where f is an eigen-function of Hecke operators in $S_{(2k+1)/2}(4N)$ (resp. $S_{(2k+1)/2}(4N, \chi_N)$) and $\Psi(f)$ is a primitive form in $S_{2k}(2N)$, where $S_{(2k+1)/2}(4N)$ (resp. $S_{(2k+1)/2}(4N, \chi_N)$) means the vector space consisting of modular cusp forms of weight $(2k+1)/2$ and of level $4N$ (resp. of level $4N$ with the character χ_N). Since the modular form f satisfying our conditions is contained in orthogonal complement of Kohnen's space, there is no overlap between Kohnen-Zagier's results and ours. The method of our proof is similar to that of [1]. To prove our results, we need to modify their methods.

§1. Notation and preliminaries. We denote by Z, Q, R and C the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $z \in C$, we put $e[z] = \exp(2\pi i z)$ and we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \leqq \pi/2$. Further, we put $z^{k/2} = (\sqrt{z})^k$ for every $k \in Z$. Let $SL(2, R)$ denote the group of all real matrices of degree 2 with determinant one and $\mathfrak{H}$ the complex upper half plane. For each positive integer N, put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R) \mid a, b, c \text{ and } d \in Z \text{ and } c \equiv 0 (\mathrm{mod}\, N) \right\}.$$

We introduce an automorphic factor $j(\gamma, z)$ of $\Gamma_0(4)$ defined by $j(\gamma, z) =$

$\theta((az+b)/(cz+d))/\theta(z)$ for every $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4)$ and for every $z\in\mathfrak{H}$ with $\theta(z)=\sum_{n=-\infty}^{\infty}e[n^2z]$.

Let k be a positive integer. We denote by $S_{(2k+1)/2}(4N)$ (resp. $S_{2k}(M)$) the vector space consisting of modular cusp forms of weight $(2k+1)/2$ (resp. $2k$) with respect to $\Gamma_0(4N)$ (resp. $\Gamma_0(M)$) (cf. [5]).

§2. Eisenstein series and the Shimura correspondence of modular forms of half integral weight. Let D and N be two square free integers such that

(1.1) $\quad D\not\equiv 2 \pmod 4,\ N\not\equiv 2 \pmod 4,\ (D,N)=1$ and $N>0$.

Define a function $\Psi_D(f)(z)$ on $\mathfrak{H}$ by

(1.2)
$$\Psi_D(f)(z)=\sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi_0(d)\Big(\frac{-1}{d}\Big)^k\Big(\frac{N|D|}{d}\Big)d^{k-1}a(N|D|(n/d)^2)\Big)e[nz]$$

for every $f(z)=\sum_{n=1}^{\infty}a(n)e[nz]\in S_{(2k+1)/2}(4N)$,

where $\chi_0(*)$ is the trivial character modulo 4 and $\Big(\frac{*}{*}\Big)$ is the quadratic residue symbol given in [5]. Then $\Psi_D(f)$ belongs to $S_{2k}(2N)$ (cf. Shimura [5] and Niwa [3]). We put $\nu(D)=(-1)^{(ND+1)/2}$. In the remainder of this paper, we assume that

(1.3) $$\chi_0(m)\Big(\frac{-1}{m}\Big)^k=\Big(\frac{4\nu(D)\operatorname{sgn}(D)}{m}\Big) \text{ for every } m\in Z.$$

We put $\tilde{D}=4\nu(D)ND$. Introduce an Eisenstein series $G_{k,\tilde{D}}(z)$ defined by

$$G_{k,\tilde{D}}(z)=(1/2)L_{\tilde{D}}(1-k)\Big(\sum_{\gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}\in\Gamma_\infty\backslash\Gamma_0(|\tilde{D}|)}\Big(\frac{\tilde{D}}{d}\Big)(cz+d)^{-k}\Big)$$

(1.4) $$=(1/2)L_{\tilde{D}}(1-k)+\sum_{n=1}^{\infty}\Big(\sum_{d|n}\Big(\frac{\tilde{D}}{d}\Big)d^{k-1}\Big)e[nz]$$

with $L_{\tilde{D}}(s)=\sum_{n=1}^{\infty}\Big(\frac{\tilde{D}}{n}\Big)n^{-s}$ and $\Gamma_\infty=\Big\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\mid n\in\boldsymbol{Z}\Big\}$.

Let D_1 and D_2 be two integers satisfying $D_2\equiv 1 \pmod 4$ and $D=D_1D_2$ and put $\tilde{D}_1=4\nu(D)ND_1$ and $\tilde{D}_2=D_2$. For $\tilde{D}_1$ and $\tilde{D}_2$, we define a function $G_{k,\tilde{D}_1,\tilde{D}_2}(z)$ on $\mathfrak{H}$ by

(1.5) $$G_{k,\tilde{D}_1,\tilde{D}_2}(z)=\sum_{n=0}^{\infty}\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(n)e[nz]$$

$$\text{with } \sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(n)=\begin{cases} -L_{\tilde{D}_1}(1-k)L_{\tilde{D}_2}(0) & \text{if } n=0, \\ \displaystyle\sum_{\substack{d_1,d_2>0 \\ d_1d_2=n}}\Big(\frac{\tilde{D}_1}{d_1}\Big)\Big(\frac{\tilde{D}_2}{d_2}\Big)d_1^{k-1} & \text{otherwise.} \end{cases}$$

For $m\in Z$, we put

$U_m(f)(z)=(1/m)\sum_{\nu \bmod |m|}f((z+\nu)/m)$ for every function $f(z)=\sum_{n=0}^{\infty}a(n)e[nz]$ on $\mathfrak{H}$.

The following lemma plays useful and important roles for our later arguments (cf. Kohnen-Zagier [1, pp. 194-196]).

Lemma 1. *The notation being as above, the transformation formulas hold*

(i) $(4N|D_1|z+1)^{-k-1/2}G_{k,\tilde{D}}(z/(4|D_1|Nz+1))\theta(N|D|z/(4|D_1|Nz+1))$

$$= \left(\frac{D_2}{-|D_1|N}\right)|D_2|^{-1}G_{k,\tilde{D}_1,\tilde{D}_2}((z+4^*N^*|D_1|^*)/|D_2|)$$
$$\theta((N|D_1|z+4^*)/|D_2|)$$

(ii) $\sum_{\substack{d|4\\ d>1}}\sum_{\mu \bmod |D_2|}(dN|D_1|(z+\mu)+1)^{-2k}G_{k,\tilde{D}}(z/(dN|D_1|(z+\mu)+1))^2$

$$= \left(\frac{D_2}{-1}\right)U_{2|D_2|}(G_{k,\tilde{D}_1,\tilde{D}_2}(z)^2),$$

where a^* *is an integer with* $aa^* \equiv 1 (\mathrm{mod}\,|D_2|)$.

Define a function $\mathscr{G}_D(z)$ on $\mathfrak{H}$ by

(1.6) $$\mathscr{G}_D(z) = \sum_{\gamma\in\Gamma_0(4|D|N)\backslash\Gamma_0(4N)} j(\gamma, z)^{-(2k+1)}G_{k,\tilde{D}}(\gamma(z))\theta(N|D|\gamma(z)).$$

Observe that $\mathscr{G}_D(z)$ becomes a modular form of weight $(2k+1)/2$ with respect to $\Gamma_0(4N)$. We can easily check

$$\Gamma_0(4|D|N)\backslash\Gamma_0(4N) = \left\{\begin{pmatrix}1 & 0\\ 4N|D_1| & 1\end{pmatrix}\begin{pmatrix}1 & \mu\\ 0 & 1\end{pmatrix}\middle| \begin{array}{l}D=D_1D_2,\ D_2\equiv 1\ (\mathrm{mod}\,4)\\ \text{and } \mu \bmod |D_2|\end{array}\right\}.$$

Hence, using Lemma 1, we may derive the following lemma.

Lemma 2. *The notation being as above,* $\mathscr{G}_D(z)$ *has the expression*

$$\mathscr{G}_D(z) = \sum_{(D_1,D_2)\in W(D)}\left(\frac{D_2}{-|D_1|N}\right)U_{|D_2|}(G_{k,\tilde{D}_1,\tilde{D}_2})(z)\theta(N|D_1|z),$$

where $W(D)=\{(D_1, D_2)\in \boldsymbol{Z}^2 \mid D=D_1D_2$ *and* $D_2\equiv 1(\mathrm{mod}\,4)\}$.

We also introduce a function $\mathscr{F}_D(z)$ on $\mathfrak{H}$ defined by

(1.7) $$\mathscr{F}_D(z) = \sum_{\gamma=\begin{pmatrix}* & *\\ c & d\end{pmatrix}\in\Gamma_0(4N|D|)\backslash\Gamma_0(2N)}(cz+d)^{-2k}G_{k,\tilde{D}}(\gamma(z))^2.$$

Now we can prove the following theorem.

Theorem 1. *Suppose that the conditions* (1.1) *and* (1.3) *are satisfied. Then* $\Psi_D(\mathscr{G}_D(z))$ *is equal to* $\mathscr{F}_D(z)$.

Proof. We put

$$\mathscr{G}_D(z) = \sum_{m=0}^{\infty} c(m)e[mz] \text{ and } \Psi_D(\mathscr{G}_D(z)) = \sum_{n=0}^{\infty} C_D(n)e[nz].$$

By Lemma 2, we have

$$c(m) = \sum_{(D_1,D_2)\in W(D)}\left(\frac{D_2}{-|D_1|N}\right)\sum_{x\in\boldsymbol{Z}}\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(m|D_2|-N|D_1|x^2)$$

and

(1.8) $$c(n^2N|D|) = \sum_{(D_1,D_2)\in W(D)}\left(\frac{D_2}{-1}\right)\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2n|D_2|}}\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(a_1a_2),$$

which yields

$$C_D(n) = \sum_{d|n}\left(\frac{4\nu(D)ND}{d}\right)d^{k-1}c((n/d)^2N|D|)$$
$$= \sum_{(D_1,D_2)\in W(D)}\left(\frac{D_2}{-1}\right)\sum_{d|n}\left(\frac{4\nu(D)DN}{d}\right)d^{k-1}\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2(n/d)|D_2|}}\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(a_1a_2)$$
$$= \sum_{(D_1,D_2)\in W(D)}\left(\frac{D_2}{-1}\right)\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=2n|D_2|}}\sum_{d|(a_1,a_2)}\left(\frac{4\nu(D)DN}{d}\right)d^{k-1}\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(a_1a_2/d^2)$$

$$= \sum_{(D_1,D_2)\in W(D)} \left(\frac{D_2}{-1}\right) \sum_{\substack{a_1,a_2\geqq 0\\ a_1+a_2=2n|D_2|}} \sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(a_1)\sigma_{k-1,\tilde{D}_1,\tilde{D}_2}(a_2).$$

We can choose the following set as a set of representatives for $\Gamma_0(4N|D|)\backslash\Gamma_0(2N)$.

$$\left\{\begin{pmatrix}1 & 0\\ dN|D_1| & 1\end{pmatrix}\begin{pmatrix}1 & \mu\\ 0 & 1\end{pmatrix}\middle| \begin{array}{l} D_1D_2 = D,\ D_2 \equiv 1 \pmod 4,\\ \mu \bmod |D_2| \text{ and } d\,|\,4(d>1)\end{array}\right\}.$$

Consequently, by Lemma 1, we conclude our assertion.

§3. Rankin's convolutions and Fourier coefficients of modular forms of half integral weight. Let $F(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ be a primitive form of $S_{2k}(2N)$ in the sence of Atkin-Lehner. The Petersson inner product of F and $\mathscr{F}_D$ is determined by

$$\langle F, \mathscr{F}_D\rangle = \int_{\Gamma_0(2N)\backslash\mathfrak{H}} F(z)\overline{\mathscr{F}_D(z)}\, y^{2k-2} dxdy. \tag{2.1}$$

The definition of $\mathscr{F}_D$ implies

$$\langle F, \mathscr{F}_D\rangle = (1/2)L_{\tilde{D}}(1-k) \int_{\Gamma_\infty\backslash\mathfrak{H}} F(z)\overline{G_{k,\tilde{D}}(z)}\, y^{2k-2} dxdy. \tag{2.2}$$

Determining of Rankin's convolution of $F(z)$ and $G_{k,\tilde{D}}(z)$, we may derive the following lemma.

Lemma 3. *Let $F(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ be a primitive form of $S_{2k}(2N)$. Then*

$$\langle F, \mathscr{F}_D\rangle = (1/2)(2k-2)!(4\pi)^{-2k+1}L_{\tilde{D}}(1-k)L(F, 2k-1)L(F, \tilde{D}, k)L_{\tilde{D}}(k)^{-1},$$

where $L(F, s) = \sum_{n=1}^{\infty} A(n)n^{-s}$ and $L(F, \tilde{D}, s) = \sum_{n=1}^{\infty}\left(\frac{\tilde{D}}{n}\right)A(n)n^{-s}$. We also confirm the following lemma (cf. Shimura [5, Theorem 1.9]).

Lemma 4. *Let $f(z) = \sum_{n=1}^{\infty} a(n)e[nz] \in S_{(2k+1)/2}(4N)$ be an eigenfunction of the Hecke operator $T^{4N}_{(2k+1)/2}(p^2)$ on $S_{(2k+1)/2}(4N)$ for each prime p, i.e.,*

$$T^{4N}_{(2k+1)/2}(p^2)f = \omega(p)f \quad \text{for every prime } p.$$

Then,

$$\langle f, \mathscr{G}_D\rangle = \Gamma(k-1/2)(4\pi N|D|)^{-k+1/2}L_{\tilde{D}}(1-k)a(N|D|)L(\tilde{F}, 2k-1)L_{\tilde{D}}(k)^{-1},$$

where $\langle f, \mathscr{G}_D\rangle = \int_{\Gamma_0(4N)\backslash\mathfrak{H}} f(z)\overline{\mathscr{G}_D(z)}y^{k-3/2}dxdy$ and $L(\tilde{F}, s) =$
$\prod_p (1-\omega(p)p^{-s} + \chi_0^2(p)p^{2k-1-2s})^{-1}$.

Let $F_i(z)\,(1 \leqq i \leqq l)$ be the set of all primitive forms in $S_{2k}(2N)$. By Ueda [7, Corollary] and Shimura [6], we can prove the following properties under the condition $k \geqq 2$:

There are a bijective linear mapping $\tilde{\Psi}$ of $S_{(2k+1)/2}(4N)$ to $S_{2k}(2N)$ and an orthogonal basis $\{f_i\}_{i=1}^{n}$ of $S_{(2k+1)/2}(4N)$ such that

(2.3) $T^{2N}_{2k}(p)\tilde{\Psi} = \tilde{\Psi}\, T^{4N}_{(2k+1)/2}(p^2)$ for every prime $p((p, 2N)=1)$, $T^{4N}_{(2k+1)/2}(p^2)f_i = \omega_i(p)f_i\,(1 \leqq i \leqq n)$ for all primes $p(p \nmid 2N)$, $T^{4N}_{(2k+1)/2}(p^2)f_i = \omega_i(p)f_i\,(1 \leqq i \leqq l)$ for all primes $p(p\,|\,2N)$, $F_i = \tilde{\Psi}(f_i)\,(1 \leqq i \leqq n)$ and $\langle F_i, F_j\rangle = 0\,(1 \leqq i \leqq l,\ l+1 \leqq j \leqq n)$,

where $T^{2N}_{2k}(p)$ (resp. $T^{4N}_{(2k+1)/2}(p^2)$) denotes the Hecke operator on $S_{2k}(2N)$ (resp. $S_{(2k+1)/2}(4N)$) for each prime p. We remark that Niwa [4] proved affirmatively the above assertion in the case where $N=1$. By the above property, $\mathcal{G}_D(z)$ has an orthogonal decomposition

$$\mathcal{G}_D(z)=\sum_{i=1}^{n}\alpha_i f_i+E(z), \tag{2.4}$$

where $E(z)$ is a linear combination of Eisenstein series of weight $(2k+1)/2$ with respect to $\Gamma_0(4N)$. By Theorem 1 and Shimura [5, Theorem 1.9], we get

$$\mathcal{F}_D(z)=\sum_{i=1}^{l}\alpha_i a_i\ (N|D|)F_i+\sum_{i=l+1}^{n}\alpha_i\Psi_D(f_i)(z)+\Psi_D(E(z)) \tag{2.5}$$

with $f_i(z)=\sum_{n=1}^{\infty}a_i(n)e[nz]$. Since $\Psi_D(f_i)\,(i\geqq l+1)$ and $\Psi_D(E(z))$ are orthogonal to the elements $\{F_i\}_{i=1}^{l}$, we have

$$\langle F_i,\mathcal{F}_D\rangle=\overline{a_i(N|D|)\langle F_i,F_i\rangle\langle f_i,\mathcal{G}_D\rangle(\langle f_i,f_i\rangle)^{-1}}\ (1\leqq i\leqq l).$$

Consequently, by Lemmas 3 and 4, we conclude the following theorem.

Theorem 2. *Suppose that* (1.1) *and* (1.3) *are satisfied* $L(F_i,2k-1)\neq 0\,(1\leqq i\leqq l)$ *and* $k\geqq 2$. *Then*

$$|a_i(N|D|)|^2=(1/2)(2k-2)!(4\pi N|D|)^{k-1/2}((4\pi)^{2k-1}\Gamma(k-1/2))^{-1}$$
$$L(F_i,\tilde{D},k)\langle f_i,f_i\rangle\langle F_i,F_i\rangle^{-1}(1\leqq i\leqq l).$$

Let χ_N be the character defined by $\chi_N(n)=\left(\frac{N}{n}\right)$ for every $n\in Z$. We denote by $S_{(2k+1)/2}(4N,\chi_N)$ the vector space consisting of modular cusp forms of Neben-type χ_N and of weight $(2k+1)/2$ with respect to $\Gamma_0(4N)$. By Ueda [7], there exist $\tilde{f}_i(1\leqq i\leqq l)$ such that

$$T^{4N}_{(2k+1)/2,\chi_N}(p^2)\tilde{f}_i=\tilde{\omega}_i(p)\tilde{f}_i \text{ for every } i\ (1\leqq i\leqq l)$$

and for every prime $p((p,4N)=1)$, where $T^{4N}_{(2k+1)/2,\chi_N}(p^2)$ is the Hecke operator on $S_{(2k+1)/2}(4N,\chi_N)$ given in Shimura [5], $\tilde{\omega}_i(p)$ is the same as in (2.3) and l is the number of all primitive forms in $S_{2k}(2N)$. Put $\tilde{f}_i(z)=\sum_{n=1}^{\infty}\tilde{a}_i(n)e[nz]$. By Theorem 2, Shimura [5, Proposition 1.5], [6] and Ueda [7], we conclude the following theorem.

Theorem 3. *The notation being as above, the assumption being as in Theorem* 2, *let* D *and* D' *be square free integers satisfying* (1.1) *and* (1.3). *Suppose that* $\tilde{a}_i(|D'|)$ *does not vanish. Then*

$$|\tilde{a}_i(|D|)/\tilde{a}_i(|D'|)|^2=(|D|/|D'|)^{k-1/2}(L(F_i,\tilde{D},k)/L(F_i,\tilde{D}',k)).$$

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