62. *Some Mean Squares in Connection with* $\zeta(1+it)$

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§1. Statement of results. The mean square of the Riemann zeta-function $\zeta(s)$ on the line $\Re s = 1$ was first considered by R. Balasubramanian et al. [1], and A. Ivić [2] obtained the formula for the higher power moment

$$\int_1^T |\zeta(1+it)|^{2k} dt = T \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2} + O((\log T)^{k^2}) \tag{1.1}$$

for any fixed natural number $k \geq 2$ and

$$\begin{aligned}\int_1^T |\zeta(1+it)|^{2k} dt = T \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2} &+ O((\log T)^{5k^2/3}(\log\log T)^{k^2/3}) \\ &+ O((\log T)^{(10k-2)/3}(\log\log T)^{(2k-1)/3}) \\ &+ O((\log T)^{(5k+3)/3}(\log\log T)^{k/3})\end{aligned} \tag{1.2}$$

for any fixed positive number k, where $d_k(n)$ is a multiplicative function defined by

$$d_k(p^m) = \binom{k+m-1}{m}$$

for arbitrary prime power $n = p^m$.

It is well-known that $\zeta(s)$ has no zeros on the line $\Re s = 1$ so that one can put the negative power moment in question. The author [3] proved

$$\int_0^T \frac{1}{|\zeta(1+it)|^{2k}} dt = T \sum_{n=1}^{\infty} \frac{\mu_k^2(n)}{n^2} + O((\log T)^{5k^2/3}(\log\log T)^{k^2/3}) \tag{1.3}$$

for any fixed natural number k, where $\mu_k(n) = d_{-k}(n)$.

In this note we study some other mean squares in connection with $\zeta(1+it)$.

Theorem. *We have*

$$\int_1^T |\zeta^{(\nu)}(1+it)|^2 dt = \zeta^{(2\nu)}(2)T + O((\log T)^{2\nu+1}) \tag{1.4}$$

for any fixed natural number ν,

$$\int_1^T \left| \frac{\zeta'(1+it)}{\zeta(1+it)} \right|^2 dt = T \sum_{n=1}^{\infty} \frac{\Lambda^2(n)}{n^2} + O((\log T)^{10/3}(\log\log T)^{2/3}), \tag{1.5}$$

$$\int_1^T |\log \zeta(1+it)|^2 dt = T \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{(n\log n)^2} + O(\log\log T) \tag{1.6}$$

and

$$\begin{aligned}\int_0^T \frac{1}{|\zeta(1+it)|^{2k}} dt = T \sum_{n=1}^{\infty} \frac{\mu_k^2(n)}{n^2} &+ O((\log T)^{5k^2/3}(\log\log T)^{k^2/3}) \\ &+ O((\log T)^{(5k+3)/3}(\log\log T)^{k/3})\end{aligned} \tag{1.7}$$

for any fixed positive number k, where $\zeta^{(\nu)}(s)$ *and* $\Lambda(n)$ *denotes the* ν*-th derivative of* $\zeta(s)$ *and the Mangoldt function respectively.*

The proof is similar to those of [1], [2] and [3] which are based on the approximate functional equation. But recently Prof. A. Ivić and Prof. K. Ramachandra kindly sent the author the letters (April and June, 1993) which say that R. Balasubramanian, A. Ivić and K. Ramachandra have also made researches on the same problem. They claim that, as for the error terms of (1.5) and (1.7), they succeeded in obtaining much better estimates than ours.

§2. Proof of the theorem. We give only the proof of (1.4) in detail. The expression

$$\zeta^{(\nu)}(s) = (-1)^{\nu} \sum_{n=1}^{\infty} (\log n)^{\nu} n^{-s} \quad (\Re s > 1) \tag{2.1}$$

and the partial summation

$$\begin{aligned} \sum_{n \le X} (\log n)^{\nu} n^{-s} &= [X](\log X)^{\nu} X^{-s} - \int_1^X [x](\log x)^{\nu-1}(\nu - s\log x)x^{-s-1}dx \\ &= \int_1^X (\log x)^{\nu} x^{-s}dx - (X-[X])(\log X)^{\nu}X^{-s} \\ &\quad + \int_1^{X} (x-[x])(\log x)^{\nu-1}(\nu - s\log x)x^{-s-1}dx \end{aligned} \tag{2.2}$$

gives an approximate functional equation

$$\zeta^{(\nu)}(s) = (-1)^{\nu} \sum_{n \le X} (\log n)^{\nu} n^{-s} + \nu(X, s) + \Xi_{(\nu)}(X, s) \tag{2.3}$$

for $\Re s > 1$, where

$$\nu(X, s) \equiv (-1)^{\nu} \int_X^{\infty} (\log x)^{\nu} x^{-s} dx = (-1)^{\nu} X^{1-s} \sum_{l=0}^{\nu} \frac{\nu!(\log X)^{\nu-l}}{(\nu-l)!(s-1)^{l+1}} \tag{2.4}$$

and

$$\begin{aligned} \Xi_{(\nu)}(X, s) \equiv {} & (-1)^{\nu}(X-[X])(\log X)^{\nu}X^{-s} \\ & + (-1)^{\nu}\int_X^{\infty} (x-[x])(\log x)^{\nu-1}(\nu - s\log x)x^{-s-1}dx. \end{aligned} \tag{2.5}$$

The functions $\nu(X, s)$ and $\Xi_{(\nu)}(X, s)$ are continued analytically for $\Re s > 0$, and $\Xi_{(\nu)}(X, 1+it) \ll tX^{-1}(\log X)^{\nu}$.

Hence

$$\begin{aligned} \int_1^T |\zeta^{(\nu)}(1+it)|^2 dt = {} & \int_1^T |\textstyle\sum_{n \le X} (\log n)^{\nu} n^{-1-it}|^2 dt \\ & + 2\Re \int_1^T \textstyle\sum_{n \le X} (\log n)^{\nu} n^{-1-it} \, \overline{\nu(X, 1+it)} \, dt \\ & + 2\Re \int_1^T \textstyle\sum_{n \le X} (\log n)^{\nu} n^{-1-it} \, \overline{\Xi_{(\nu)}(X, 1+it)} \, dt \\ & + 2\Re \int_1^T \nu(X, 1+it) \, \overline{\Xi_{(\nu)}(X, 1+it)} \, dt + \int_1^T |\nu(X, 1+it)|^2 dt \\ & + \int_1^T |\Xi_{(\nu)}(X, 1+it)|^2 dt = \sum_{i=1}^{6} \int_i, \ \textit{say}. \end{aligned} \tag{2.6}$$

By the Montgomery-Vaughan theorem,

$$(2.7)\quad \int_1 = (T-1)\sum_{n\le X}(\log n)^{2\nu}n^{-2} + O\Big(\sum_{n\le X}(\log n)^{2\nu}n^{-1}\Big)$$
$$= T\sum_{n=1}^{\infty}(\log n)^{2\nu}n^{-2} + O\Big(T\sum_{n>X}(\log n)^{2\nu}n^{-2}\Big) + O\Big((\log X)^{2\nu}\sum_{n\le X}n^{-1}\Big)$$
$$= \zeta^{(2\nu)}(2)T + O\Big(TX^{-1/2}\sum_{n>X}(\log n)^{2\nu}n^{-3/2}\Big) + O((\log X)^{2\nu+1}).$$

To evaluate the second integral we use the similar way as in [1], such that

$$(2.8)\quad \int_2 \ll (\log X)^{2\nu}\sum_{n\le X}n^{-1}\int_1^T (X/n)^{it}t^{-1}dt$$
$$= (\log X)^{2\nu}\sum_{n\le X(1-1/\log X)}(n\log(X/n))^{-1} + (\log X)^{2\nu}\log T\sum_{X(1-1/\log X)<n\le X}n^{-1}$$
$$\ll (\log X)^{2\nu}\log\log X + (\log X)^{2\nu-1}\log T.$$

The remaining integrals are immediately evaluated by

$$T^2X^{-1}(\log X)^{2\nu+1},\ (\log X)^{2\nu},\ TX^{-1}(\log X)^{2\nu},\ T^3X^{-2}(\log X)^{2\nu}$$

respectively. The desired formula (1.4) is obtained by putting $X = T^2$.

To prove the formula (1.5) and (1.6) we start from the classical results

$$(2.9)\quad \sum_{n\le X}\Lambda(n) = X + O(X\exp(-c(\log X)^{3/5}(\log\log X)^{-1/5}))$$

$$(2.10)\quad \sum_{2\le n\le X}\frac{\Lambda(n)}{\log n} = \int_2^X\frac{du}{\log u} + O(X\exp(-c(\log X)^{3/5}(\log\log X)^{-1/5}))$$

and the approximate functional equations are given by

$$(2.11)\quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\le X}\Lambda(n)n^{-s} - \frac{X^{1-s}}{1-s} + \Xi_\phi(X, s),$$

$$(2.12)\quad \log\zeta(s) = \sum_{2\le n\le X}\frac{\Lambda(n)}{\log n}n^{-s} + \int_X^\infty\frac{du}{u^s\log u} + \Xi_\Pi(X, s)$$

respectively, where the error terms $\Xi_\phi(X, s)$ and $\Xi_\Pi(X, s)$ are just like $\Xi_k(X, s)$ in [3]. To prove (1.7) we require an appropriate estimation of some integral which is a slight modification of [2].

References

[1] R. Balasubramanian, A. Ivić and K. Ramachandra: The mean square of the Riemann zeta-function on the line $\sigma = 1$. L'Enseignement Math., **38**, 13–25 (1992).

[2] A. Ivić: The moments of the zeta-function on the line $\sigma = 1$ (preprint).

[3] H. Nakaya: The negative power moment of the Riemann zeta-function on the line $\sigma = 1$ (preprint).