60. *Value Groups of Henselian Valuations*

By Tsuneo KANNO*) and Takeo SHIRASOU**)

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 13, 1993)

0. Introduction. Let us begin with Neukirch's formulation of general class field theory ([1], [2]). Let G be a pro-finite group and let (G_K) be the closed subgroups of G indexed by "fields" K. Take the "ground field" k such that $G = G_k$. For fields L and K, L is called an extension of K denoted by L/K, if G_K contains G_L and the group index $[G_K : G_L]$ is called the extension degree of L/K denoted by $[L : K]$. Further, if G_L is a normal subgroup of G_K, L is a Galois extension of K with the Galois group $G(L/K) = G_K/G_L$.

For fields K_1 and K_2, the composite field $K_1 K_2$ is defined to be a field such that $G_{K_1K_2} = G_{K_1} \cap G_{K_2}$, and the intersection $K_1 \cap K_2$ is defined to be a field such that $G_{K_1 \cap K_2}$ is the closed subgroup of G generated topologically by G_{K_1} and G_{K_2}.

Let $\hat{\boldsymbol{Z}}$ be the completion of the module $\boldsymbol{Z}$ of rational integers with respect to the finite-index-subgroup-topology. Take a surjective continuous homomorphism deg: $G_k \to \hat{\boldsymbol{Z}}$ and let $\tilde{k}$ be a field such that $G_{\tilde{k}}$ is the kernel of deg. For a finite extension K of k, put $\tilde{K} = K\tilde{k}$ and $f_K = [K \cap \tilde{k} : k]$.

Now suppose that a multiplicative G-module A is given. For a field K let A_K be the submodule of A of elements fixed by G_K. And for a finite extension L of K, we have a homomorphism $N_{L|K} : A_L \ni a \to \prod_{\sigma \in G_K/G_L} a^{\sigma} \in A_K$.

In [2], Neukirch defined a *Henselian valuation* with respect to deg to be a homomorphism $v : A_k \to \hat{\boldsymbol{Z}}$ satisfying the following two conditions;

(i) the image $Z = v(A_k)$ contains $\boldsymbol{Z}$ and $\boldsymbol{Z}/n\boldsymbol{Z} \simeq Z/nZ$ for any positive integer n,

(ii) $v(N_{K|k} A_K) = f_K \cdot Z$ for any finite extension K of k.

In this paper, any family $(G, A, \deg, v)$ as above will be called an *admissible situation* over k.

We shall study here the structure of the value group Z of a Henselian valuation v and show that if for any finite subextension L/K of $\tilde{K}/K$ the class field axiom

$$^{\#}H^i(G(L/K), A_L) = \begin{cases} [L:K] & \text{if } i = 0 \\ 1 & \text{if } i = -1 \end{cases}$$

holds, then a Henselian valuation v is essentially determined by G, A and deg.

Neukirch has shown that an admissible situation $(G, A, \deg, v)$ gives a "class field theory", if the class field axiom holds for any finite cyclic extension L/K. Thus our result will show that a Henselian valuation v is essentially unique in Neukirch's class field theory.

---

*) Department of Mathematics, Josai University.
**) Department of Mathematics, Utsunomiya University.

Our theorem and its Corollary 1 can be proved by using a property of the norm residue symbol after establishing the class field theory (cf. [3]). Note that our proofs being based on an elementary property of the value group Z, the results follow directly from the definition of $(G, A, \deg, v)$ and the class field axiom for special kind of field extensions.

The authors thank Neukirch for his profitable comments.

1. Certan kind of submoules of $\hat{\boldsymbol{Z}}$. In this section we shall study modules as defined below.

Definition. A submodule Z of $\hat{\boldsymbol{Z}}$ is a *module of type* (v) if Z contains $\boldsymbol{Z}$ and satisfies $\boldsymbol{Z}+nZ=Z$ for any positive integer n.

$\boldsymbol{Z}$ and $\hat{\boldsymbol{Z}}$ are modules of type (v).

As for the first condition of the definition, we have

Proposition 1. *If a submodule Z of $\hat{\boldsymbol{Z}}$ contains $\boldsymbol{Z}$. Then for any positive integer n we have*

$$\boldsymbol{Z}\cap nZ=n\boldsymbol{Z},$$

and the natural homomorphism $\rho:\boldsymbol{Z}/n\boldsymbol{Z}\rightarrow Z/nZ$ is injective.

Proof. Take $z=(z_p)\in\hat{\boldsymbol{Z}}=\Pi_p\boldsymbol{Z}_p$ such that $n\cdot z=m\in\boldsymbol{Z}$, where $\boldsymbol{Z}_p$ is the p-adic integer ring. Then we have $n\cdot z_p=m$ for any prime p. Considering p-adic values of both sides we have that n is a divisor of m. Hence we get $(\boldsymbol{Z}\cap n\hat{\boldsymbol{Z}})=n\boldsymbol{Z}$. Now we have, for Z as above, $n\boldsymbol{Z}\subset(\boldsymbol{Z}\cap nZ)\subset(\boldsymbol{Z}\cap n\hat{\boldsymbol{Z}})=n\boldsymbol{Z}$ and $(\boldsymbol{Z}\cap nZ)=n\boldsymbol{Z}$. Q.E.D.

Corollary. *For a submodule Z of $\hat{\boldsymbol{Z}}$ containing $\boldsymbol{Z}$ and a positive integer n the following conditions are equivalent to each other*;

(i) $\boldsymbol{Z}+nZ=Z$

(ii) *Z/nZ is of order n*

(iii) *the natural homomorphism ρ is an isomorphism $\boldsymbol{Z}/n\boldsymbol{Z}\simeq Z/nZ$.*

Hence the condition (i) of the Henselian valuation is equivalent to the condition that the value group Z is a module of type (v).

Now we shall construct a module of type (v) which is different from $\boldsymbol{Z}$ and $\hat{\boldsymbol{Z}}$. For any rational prime p, let $\boldsymbol{Z}_{(p)}$ be the localization of $\boldsymbol{Z}$ at the prime ideal $p\boldsymbol{Z}$ i.e. $\boldsymbol{Z}_{(p)}=\{m/n\mid m,n\in\boldsymbol{Z},\ (n,p)=1\}$. Then we can embed $\boldsymbol{Z}_{(p)}$ in $\boldsymbol{Z}_p$ and we have a subring $Z=\Pi\,\boldsymbol{Z}_{(p)}$ of $\hat{\boldsymbol{Z}}=\Pi\,\boldsymbol{Z}_p$.

Proposition 2. *The above $Z=\Pi_p\boldsymbol{Z}_{(p)}$ is a module of type (v).*

Proof. We have $\boldsymbol{Z}\subsetneq Z\subsetneq\hat{\boldsymbol{Z}}$. It is sufficient to show that for any positive integer n, $Z\subset(\boldsymbol{Z}+nZ)$. Take $z=(z(q))\in Z=\Pi_q Z_{(q)}$.

In the case $n=p^e$, write $z(p)=m/n$ for some m and n such that $(n,p)=1$. Taking integers a and b such that $an+bp^e=1$, we have $z(p)=m(an+bp^e)/n=s+p^e\cdot z'(p)$ where $s=ma\in\boldsymbol{Z}$ and $z'(p)=mb/n\in Z_{(p)}$. For a prime $q\neq p$, let $z'(q)=(z(q)-s)/p^e\in\boldsymbol{Z}_{(q)}$, then we have $z(q)=s+p^e\cdot z'(q)$. Further, put $z'=(z'(q))\in Z$ and $z=s+p^e\cdot z'$.

$Z\subset(\boldsymbol{Z}+nZ)$ for general n follows from the fact that for any submodule Z and any relatively prime integers m and n we have

$$(Z+mZ)\cap(\boldsymbol{Z}+nZ)\subset(\boldsymbol{Z}+mnZ).$$ Q.E.D.

Of course, we have submodules of $\hat{\boldsymbol{Z}}$ containing $\boldsymbol{Z}$ which are not modules of type (v) e.g. $Z_1=\Pi\,\boldsymbol{Z}$ and $Z_2=\boldsymbol{Z}+\boldsymbol{Z}_p$.

Proposition 3. *Let Z and Z' be modules of type (v). Then for any isomorphism $\sigma : Z \stackrel{\sim}{\rightarrow} Z'$ there exists a unit u of $\hat{\boldsymbol{Z}}$ such that $u^{-1} \in Z$, $u \in Z'$ and $\sigma(z) = uz$ for all $z \in Z$.*

Conversely, for any unit u_1 of $\hat{\boldsymbol{Z}}$ such that $u_1^{-1} \in Z$, the set $u_1 Z = \{u_1 z \mid z \in Z\}$ is a module of type (v) and the mapping: $z \rightarrow u_1 z$ is an isomorphism $Z \stackrel{\sim}{\rightarrow} u_1 Z$.

Proof. Considering $1 \in \boldsymbol{Z} \subset Z$, put $u = \sigma(1) \in Z'$. Then $\sigma(m) = um$ for any $m \in \boldsymbol{Z}$. Take any $z \in Z$ which is a module of type (v). Since for any positive integer n we have $\boldsymbol{Z} + nZ = Z$, we can write $z = m + nz_1$ for some $m \in \boldsymbol{Z}$ and $z_1 \in Z$. Then we have

$$\sigma(z) = \sigma(m) + n \cdot \sigma(z_1) = um + unz_1 - unz_1 + n\sigma(z_1) = uz + n(\sigma(z_1) - uz_1).$$

Hence we have $\sigma(z) - uz \in n\hat{\boldsymbol{Z}}$ for any positive integer n. Now from $\bigcap_n n\hat{\boldsymbol{Z}} = \{0\}$ follows $\sigma(z) = uz$ for all $z \in Z$.

The remaining part is easily seen. Q.E.D.

From the proposition follows

Corollary. *If a module Z of type (v) is isomorphic to $\boldsymbol{Z}$ (or $\hat{\boldsymbol{Z}}$), then $Z = \boldsymbol{Z}$ (or $\hat{\boldsymbol{Z}}$, respectively).*

2. Henselian valuations. Here is an elementary key lemma;

Lemma. *Let M be an additive group and let n be a positive integer. Then if the subgroup nM is of index n, nM is the unique subgroup of M of index n.*

The value group Z of a Henselian valuation has the unique subgroup nZ of index n for any positive integer n.

Theorem. *Let $(G, A; \deg, v)$ be an admissible situation over k. For a finite extension K of k, let*

$$v_K = (1/f_K)\, v \circ N_{K/k} : A_K \rightarrow \hat{\boldsymbol{Z}}$$

and put $U_K = \{u \in A_K \mid v_K(u) = 0\}$, called the unit group of v_K. Then, if for finite subextension L/K of $\tilde{K}/K$ the class field axiom holds, we have

$$N_{L|K}(A_L) = A_K^n\, U_K,$$

where $n = [L : K]$ is the extension degree.

Proof. Let $Z = v(A_k)$, then Z is also the image of v_K. Hence we have $A_K / U_K \simeq Z$. Further the isomorphism induces $A_K^n U_K / U_K \simeq nZ$. Then the above lemma tells that $A_K^n U_K / U_K$ is the unique subgroup of A_K / U_K of index n and that $A_K^n U_K$ is the unique subgroup of A_K of index n containing U_K.

On the other hand, from the class field axiom follows that the group $A_K / N_{L/K}(A_L) = H^0(G(L/K), A_L)$ is of order n. Further, considering $H^0(G(L/K), U_L) = 0$ (cf. [2] p.22 Prop. 22) i.e. $U_K = N_{L/K}(U_L)$, we have $U_K \subset N_{L|K}(A_L)$. This means that $N_{L|K}(A_L)$ is also a subgroup of A_K of index n containing U_K. Since such subgroup is unique, we get $N_{L|K}(A_L) = A_K^n U_K$.

Q.E.D.

Now, as corollaries of the theorem we state exactly what is the purpose of our paper. This fact is essentially pointed out in [3].

Corollary 1. *Let $(G, A, \deg, v)$ and $(G, A, \deg, v')$ be admissible situations over k and let K be a finite extension of k. Then if the class field axiom holds for any subextension L/K of $\tilde{K}/K$ in both situations, the valuations v_K and v'_K have the same unit group.*

Proof. Since the homomorphism $\deg_K = (1/f_K)\deg : G_K \to \hat{\mathbf{Z}}$ induces $G_K/G_{\tilde{K}} \simeq \hat{\mathbf{Z}}$, for any positive integer n there exists a subextension L/K of $\tilde{K}/K$ whose Galois group is a cyclic group of order n. Hence by the theorem we get $A_K^n U_K = N_{L|K}(A_L) = A_K^n U_K'$ for any positive integer n, where U_K' is the unit group of v_K'.

On the other hand, $A_K/U_K \simeq v_K(A_K) = Z$ induces $A_K^n U_K/U_K \simeq nZ$. Hence, $(\bigcap_n A_K^n U_K)/U_K \subset \bigcap_n (A_K^n U_K/U_K) \simeq \bigcap_n nZ \subset \bigcap_n n\hat{\mathbf{Z}} = \{0\}$. Thus we have $\bigcap_n A_K^n U_K = U_K$.

Similarly we have $\bigcap_n A_K^n U_K' = U_K'$. Hence we have $U_K = U_K'$. Q.E.D.

Corollary 2. *Let notations and assumption be the same as those of Corollary 1. Then there exists a unit* $u \in \hat{\mathbf{Z}}$ *such that* $u \in v_K'(A_K)$ *and* $u^{-1} \in v_K(A_K)$ *and that* $v_K'(a) = u \cdot v_K(a)$ *for all* $a \in A_K$.

Conversely, for any unit u_1 *of* $\hat{\mathbf{Z}}$ *such that* $u_1^{-1} \in v\ (A_k)$ *let*

$$v'' : A_k \ni a \to u_1 \cdot v(a) \in \hat{\mathbf{Z}}.$$

Then $(G, A, \deg, v'')$ *is an admissible situation over* k *satisfying the class field axiom for any subextension of* $\tilde{K}/K$.

Proof. From Corollary 1 follows $v_K(A_K) \simeq v_K'(A_K)$. By the proposition 3 we get the direct part. The converse part is easily seen. Q.E.D.

Example. A Henselian valuation of the local (or global) class field theory (cf. [2]) is unique up to a multiplication by ± 1 (or a unit of $\hat{\mathbf{Z}}$, respectively).

In this connection, it may be natural to ask the following question: For a given local or global field k, let G be the Galois group of the separable closure of k over k. Can one construct a new class field theory $(G, A, \deg, v)$ such that the image of v is different from both classical value groups $\mathbf{Z}$ and $\hat{\mathbf{Z}}$, for example, such that the image is Z of Proposition 2 ?

References

[1] J. Neukirch: Neubegründung der Klassenkörpertheorie. Math. Zeit., **189**, 557–574 (1984).
[2] ——: Class Field Theory. Springer-Verlag (1985).
[3] T. Shirasou: Some property of norm residue symbol in Neukirch's general class field theory (appearing in Bull. Fac. of Education, Utsunomiya Univ.).