59. *The Restriction of $A_q(\lambda)$ to Reductive Subgroups*

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1. Discrete decomposability with respect to symmetric pairs. Let G be a real reductive linear Lie group and $\hat{G}$ the unitary dual of G. Suppose G' is a reductive subgroup of G. The representation $\pi \in \hat{G}$ is called *G'-admissible* if the restriction $\pi_{|G'}$ splits into a discrete sum of irreducible representations of G' with finite multiplicity. It may well happen that the restriction $\pi_{|G'}$ contains continuous spectrum (even worse, with infinite multiplicity) which is sometimes difficult to analyse. Thus, the notion of admissibility is emphasized here to single out a very nice pair (π, G') for the study of the restriction $\pi_{|G'}$. Here are famous examples where $\pi \in \hat{G}$ is G'-admissible.

(1.1)(a) If G' is a maximal compact subgroup of G, then any $\pi \in \hat{G}$ is G'-admissible (Harish-Chandra). An explicit decompostition formula is known as a *generalized Blattner formula* if $\pi = A_{\mathfrak{q}}(\lambda)$ (attached to elliptic orbits in the sense of orbit method; see [2], [9] Theorem 6.3.12).

(1.1)(b) A restriction formula of a holomorphic discrete series G' is found with respect to some reductive subgroups G' (eg. [7], [4]). Also the restriction of the Segal-Shale-Weil representation π with respect to dual reductive pair with one factor compact is intensively studied (Howe's correspondence).

We remark that G' is compact in the case (1.1)(a), while $\pi \in \hat{G}$ is a highest weight module in (1.1)(b). On the other hand, in some special settings, explicit restriction formulas have been found where $\pi \in \hat{G}$ does not belong to unitary highest weight modules but is G'-admissible for noncompact $G' \subset G$, such as $(G, G') \simeq (SO(4,2), SO(4,1))$ and π is non-holomorphic discrete series ([5] Example 3.4.2), $(G, G') = (SO(4,3), G_2(\boldsymbol{R}))$ and π is in some family of derived functor modules (Kobayashi-Uzawa, 1989 at Math. Soc. Japan), and a recent work of Howe and Tan [3]. See also an explicit formula of the discrete part of $\pi_{|G'}$ for $(G, G') \simeq (SO(3,2), SO(2,2))$ and π non-holomorphic discrete series in [1] in the non-admissible case. In this section we find a more general but still good framework to study the restriction $\pi_{|G'}$.

Let θ be a Cartan involution of G. Write $\mathfrak{g}_0$ for the Lie algebra of G, $\mathfrak{g} = \mathfrak{g}_0 \otimes \boldsymbol{C}$ for its complexification, $K = G^{\theta}$ for the fixed point group of θ, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ for the corresponding Cartan decomposition. Take a fundamental Cartan subalgebra $\mathfrak{h}_0^c (\subset \mathfrak{g}_0)$. Then $\mathfrak{t}_0^c := \mathfrak{h}_0^c \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. A θ-stable parabolic subalgebra $\mathfrak{q} \equiv \mathfrak{q}(\lambda) = \mathfrak{l}(\lambda) + \mathfrak{u}(\lambda) \subset \mathfrak{g}$ and a Levi part $L(\lambda) \subset G$ are given by an elliptic element $\lambda \in \sqrt{-1}(\mathfrak{t}_0^c)^*$ (see [9] Definition 5.2.1). Let $\mathscr{R}_{\mathfrak{q}}^j \equiv (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j$ ($j \in \boldsymbol{N}$) be the Zuckerman's derived functor from the category of metaplectic $(\mathfrak{l}, (L \cap K)^{\sim})$-modules to that of $(\mathfrak{g}, K)$-modules. In this paper, we follow the normalization in [10] Definition

6.20 and some terminologies such as *weakly fair* in [11] Definition 2.5.

Let σ be an involutive automorphism of G. If G' is an open subgroup of the fixed points of σ, (G, G') is called a *reductive symmetric pair.* Choose a Cartan involution θ of G so that $\sigma\theta = \theta\sigma$. Then $K' := K \cap G'$ is a maximal compact subgroup of G'. We write $\mathfrak{k}_{0\pm} := \{X \in \mathfrak{k}_0 : \sigma(X) = \pm X\}$. Fix a σ-stable Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$ such that $\mathfrak{t}_{0-}^c := \mathfrak{t}_0^c \cap \mathfrak{k}_{0-}$ is a maximal abelian subspace in $\mathfrak{k}_{0-}$. Choose a positive system $\Sigma^+(\mathfrak{k}, \mathfrak{t}_-^c)$ of the restricted root system $\Sigma(\mathfrak{k}, \mathfrak{t}_-^c)$ and a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ which is compatible with $\Sigma(\mathfrak{k}, \mathfrak{t}_-^c)$. Let $\mathfrak{q} = \mathfrak{q}(\mu) = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ given by an element $\mu \in \sqrt{-1}(\mathfrak{t}_0^c)^*$, which we can assume to be dominant with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t}^c)$ without loss of generality. Define a closed cone in $\sqrt{-1}(\mathfrak{t}_0^c)^*$ by

$$\boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle := \{ \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)} n_\beta \beta : n_\beta \geq 0 \}.$$

Theorem 1.2. *In the setting as above, if* $\boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \sqrt{-1}(\mathfrak{t}_{0-}^c)^* = \{0\}$, *then* $\overline{\mathcal{R}_{\mathfrak{q}}^S(\boldsymbol{C}_\lambda)}$ *is K'-admissible for any metaplectic unitary character* $\boldsymbol{C}_\lambda$ *of* $\tilde{L}$ *in the weakly fair range. In particular,* $\overline{\mathcal{R}_{\mathfrak{q}}^S(\boldsymbol{C}_\lambda)}$ *is G'-admissible.*

Remark 1.3. In Proposition 4.1.3 in [6], we have established a different type of admissibility in the case where $\mathfrak{k}$ has a direct sum decomposition $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, $G' \supset K_1$ and $\mathfrak{q} = \mathfrak{q}(\mu)$ such that $\mu_{|\mathfrak{t}^c \cap \mathfrak{k}_2} = 0$.

2. Discrete series for homogeneous spaces of reductive type. Let G be a Lie group and G' its closed subgroup. Then G' naturally acts on $X = G/H$ from the left. Given $x \in G/H$, we write the isotropy subgroup $H' \equiv G'_x := \{g \in G' : g \cdot x = x\}$ and put $X' = G'/H'$. As a representation theoretic counterpart of an embedding $f : X' \hookrightarrow X$ we consider the restriction of representations of G with respect to G' which arises as the pullback of function spaces $f^* : \Gamma(X) \to \Gamma(X')$.

If H is a reductive algebraic subgroup of a real reductive linear Lie group G, we say the homogeneous space G/H *of reductive type.* An irreducible unitary representation $\pi \in \hat{G}$ is called *discrete series* for $L^2(G/H)$ if π can be realized as a closed invariant subspace of $L^2(G/H)$. The totality of discrete series for $L^2(G/H)$ is denoted by $\mathrm{Disc}(G/H) (\subset \hat{G})$. We also write $\mathbf{Disc}(G/H)$ for the multiset of $\mathrm{Disc}(G/H)$ counted with multiplicity occurring in $L^2(G/H)$. Analogous notation is used for L^2-sections of G-equivariant vector bundles over G/H associated to a unitary representation of H. On the other hand, given $(\pi, V) \in \hat{G}$, we write $\mathrm{Disc}(\pi_{|H}) (\subset \hat{G})$ for the set of irreducible discrete summands of the restriction $\pi_{|H}$, and $\mathbf{Disc}(\pi_{|H})$ for the corresponding multiset counted with multiplicity.

Theorem 2.1. *Suppose G is a real reductive linear group and G', H are reductive subgroups stable under θ simultaneously. Let $H' := H \cap G'$. Assume there exists a minimal parabolic subgroup P' of G' such that*

(2.1)(a) $\dim H + \dim G' = \dim G + \dim (H \cap G')$,

(2.1)(b) $\dim H' + \dim P' = \dim G' + \dim (H' \cap P')$.

Then we have a bijection between multisets $\bigcup_{\pi \in \mathbf{Disc}(G/H)} \mathbf{Disc}(\pi_{|G'}) \simeq \mathbf{Disc}(G'/H')$. *In particular,* $\mathrm{Disc}(G'/H') = \emptyset$ *if and only if either* $\mathrm{Disc}(G/H) = \emptyset$ *or* $\pi_{|G'}$

is decomposed into only continuous spectrum for any $\pi \in \mathrm{Disc}(G/H)$. *Moreover, if discrete series for* G'/H' *is multiplicity free, then the discrete part of the restriction of* $\pi_{|G'}$ *is multiplicity free for all* $\pi \in \mathrm{Disc}(G/H) \subset \hat{G}$.

An abundant theory on the harmonic analysis on G/H has been developed in these fifteen years when G/H is a semisimple symmetric space, while very little has been studied when it is non-symmetric. We note that if one knows $\mathrm{Disc}(G/H)$ and the restriction formula $\pi_{|G'}$ for $\pi \in \mathrm{Disc}(G/H)$, then Theorem (2.1) gives a construction and exhaustion of discrete series for G'/H'. More weakly, only a combination of Theorem (1.2) and Theorem (2.1) gives new results on the existence of discrete series of some non-symmetric spherical homogeneous spaces such as

Corollary 2.2 1) $\mathrm{Disc}(SU(2p-1,2q)/Sp(p-1,q)) \neq \emptyset$ *for any* p, q.
2) $\mathrm{Disc}(SO(2p-1,2q)/U(p-1,q)) \neq \emptyset$ *if and only if* $pq \in 2\boldsymbol{Z}$.
3) $\mathrm{Disc}(SO(4,3)/G_2(\boldsymbol{R})) \neq \emptyset$, $\mathrm{Disc}(G_2(\boldsymbol{R})/SL(3,\boldsymbol{R})) \neq \emptyset$.

Now, relax the assumption (2.1)(a). In the setting at the beginning §2, we say $f : G'/H' \subset G/H$ *regular* if there exists a submanifold I of G/H such that $G'_y = H'$ for any $y \in I$ and that $\varphi : G'/H' \times I \to G/H$, $(g,y) \mapsto g \cdot y$ is an open embedding.

Example 2.3 (group manifolds). If $H' = H = \{e\}$, then $G' \subset G$ is regular. We can take I to be a local section of the principal bundle $G \to G/G'$.

Example 2.4 (semisimple orbits in symmetric spaces). Let σ, τ be commutative involutive automorphisms of G, (G, G') and (G, H) the corresponding symmetric pairs. Fix a maximally abelian semisimple subspace $\mathfrak{a}$ in $\{X \in \mathfrak{g}_0 : \sigma(X) = \tau(X) = -X\}$ and define $M' := \{g \in G' \cap H : \mathrm{Ad}(g)X = X \text{ for } X \in \mathfrak{a}\}$. Then $G'/M' \subset G/H$ is regular. The regular semisimple orbit in G under the adjoint action of G is a typical example.

Theorem 2.5. *In the setting of Theorem* (2.1), *suppose* $\varphi_j : G' \times H'_j \times I_j \to G/H$ $(j \in J)$ *define regular orbits such that the disjoint union of* $\varphi_j(G'/H'_j \times I_j)$ *is open dense in* G/H. *Then we have* $\cup_{\pi \in \mathrm{Disc}(G/H)} \mathrm{Disc}(\pi_{|G'}) \subset \cup_j \mathrm{Disc}(G'/H'_j)$. *In particular, if* $\mathrm{Disc}(G'/H'_j) = \emptyset$ $(j \in J)$, *then either* $\mathrm{Disc}(G/H) = \emptyset$ or $\mathrm{Disc}(\pi_{|G'}) = \emptyset$ *for any* $\pi \in \mathrm{Disc}(G/H)$. *Moreover, if* $\pi \in \mathrm{Disc}(G/H)$ *is* K'*-admissible, then* $\mathrm{Disc}(\pi_{|G'}) \subset \cap_j \mathrm{Disc}(G'/H'_j)$.

Here is a very special case corresponding to Example (3.2):

Corollary 2.6. *Suppose* $\pi = \overline{A_{\mathfrak{q}}} \in \hat{G}$ *is a (Harish-Chandra's) discrete series for* G. *If* π *is* G'*-admissible, then* $\pi_{|G'}$ *is decomposed into discrete series for* G', *In particular, if* $\mathrm{rank} G' > \mathrm{rank} K'$ *and* $\mathrm{rank} G = \mathrm{rank} K$, *then* $\pi_{|G'}$ *is decomposed into only continuous spectrum.*

Remark 2.7. In general, if $\pi \in \mathrm{Disc}(G)$, then $\pi_{|G'}$ is supported on tempered representations of G' by Mackey-Anh's reciprocity theorem.

3. Examples of decomposition formulas. In the framework of §1, §2 we present some explicit branching formulas joint with B.Ørsted.

Let $G = SO_0(p, q) \supset K = SO(p) \times SO(q)$ $(p \geq 1, q \geq 0)$. We take a (standard) basis $\{f_i\}$ of $\sqrt{-1}(\mathfrak{t}_0^c)^*$ as in [6] §2.5 and define θ-stable parabolic subalgebras by $\mathfrak{q}_+ := \mathfrak{q}(f_1) = \mathfrak{l} + \mathfrak{u}_+$, $\mathfrak{q}_- := \mathfrak{q}(-f_1) = \mathfrak{l} + \mathfrak{u}_-$ $(p \geq 2)$.

Then $L(f_1) = L(-f_1) \simeq \boldsymbol{T} \times SO_0(p-2, q)$. Put $Q := \frac{1}{2}(p+q) - 2$. For $\lambda \in \boldsymbol{Z} + Q$, we write $\boldsymbol{C}_{\lambda f_1}$ for the metaplectic representation of $\tilde{L}$ corresponding to $\lambda f_1 \in \sqrt{-1}(\mathfrak{t}_0^c)^*$. If $\lambda \in \boldsymbol{Z} + Q$ and $\lambda \geq 0$ (moreover if $\lambda \geq \frac{1}{2}p - 1$ when $q = 0$), we define $(\mathfrak{g}, K)$-modules by

$$U_+(\lambda) \equiv U_+^{SO_0(p,q)}(\lambda) := (\mathcal{R}_{\mathfrak{q}_+}^{\mathfrak{g}})^{p-2}(\boldsymbol{C}_{\lambda f_1}), \quad U_-(\lambda) \equiv U_-^{SO_0(p,q)}(\lambda) := (\mathcal{R}_{\mathfrak{q}_-}^{\mathfrak{g}})^{p-2}(\boldsymbol{C}_{-\lambda f_1}).$$

Then $U_\pm(\lambda)$ are non-zero irreducible $(\mathfrak{g}, K)$-modules and $U_\pm(\lambda) \in \mathrm{Disc}(SO_0(p, q)/SO_0(p-1, q))$ if $\lambda > 0$.

Next, let $G' = U(p, q) \supset K' = U(p) \times U(q)$. We represent the root system of $\mathfrak{k}'$ as $\Delta(\mathfrak{k}', \mathfrak{t}'^c) = \{\pm(e_i - e_j) : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q\}$. We define θ-stable parabolic subalgebras of $\mathfrak{g}'$ by

I) For $p \geq 1$, $q \geq 1$, $\mathfrak{q}'_+ := \mathfrak{q}(2e_1 + e_{p+1})$ and $\mathfrak{q}'_- := \mathfrak{q}(-2e_p - e_{p+q})$.

II) For $p \geq 2$, $q \geq 0$, $\mathfrak{q}'_0 := \mathfrak{q}(e_1 - e_p)$.

For $\lambda \in \boldsymbol{N}_+$, $l \in \boldsymbol{Z}$ such that $l \equiv \lambda + p + q + 1 \mod 2$, we define $(\mathfrak{g}', K')$-modules by:

$$V_+(\lambda, l) \equiv V_+^{U(p,q)}(\lambda, l) := (\mathcal{R}_{\mathfrak{q}'_+}^{\mathfrak{g}'})^{p+q-2}(\boldsymbol{C}_{\frac{\lambda+l}{2}e_1 + \frac{-\lambda+l}{2}e_{p+1}}) \quad \text{if } l > \lambda > 0,\ pq \geq 1,$$

$$V_0(\lambda, l) \equiv V_0^{U(p,q)}(\lambda, l) := (\mathcal{R}_{\mathfrak{q}'_0}^{\mathfrak{g}'})^{2p-4}(\boldsymbol{C}_{\frac{\lambda+l}{2}e_1 + \frac{-\lambda+l}{2}e_p}) \quad \text{if } \lambda \geq |l|,\ p \geq 2,$$

$$V_-(\lambda, l) \equiv V_-^{U(p,q)}(\lambda, l) := (\mathcal{R}_{\mathfrak{q}'_-}^{\mathfrak{g}'})^{p+q-2}(\boldsymbol{C}_{\frac{-\lambda+l}{2}e_p + \frac{\lambda+l}{2}e_{p+q}}) \quad \text{if } -l > \lambda > 0,\ pq \geq 1.$$

If $q \geq 1$, then we have (cf. [6] Theorem2):

(3.1)(a) $V_+(\lambda, l)$, $V_0(\lambda, l)$, $V_-(\lambda, l)$ are non-zero and irreducible $(\mathfrak{g}', K')$-modules with $Z(\mathfrak{g}')$-infinitesimal character $\left(\frac{\lambda+l}{2}, \frac{-\lambda+l}{2}, Q', Q'-1, \ldots, -Q'\right)$ in the Harish-Chandra parametrization, where $Q' := \frac{p+q-3}{2}$

(3.1)(b) $V_+(\lambda, l) \simeq V_-(\lambda, -l)^\vee \quad (l > \lambda > 0)$, $\quad V_0(\lambda, l) \simeq V_0(\lambda, -l)^\vee \quad (\lambda \geq |l|)$.

(3.1)(c) $\mathbf{Disc}(U(p, q)/U(1) \times U(p-1, q); \chi_l)$ $(q \geq 1, l \in \boldsymbol{Z})$ are given by,

$$\begin{cases} \{V_\varepsilon(\lambda, l) : |l| > \lambda > 0\} \cup \{V_0(\lambda, l) : \lambda \geq |l|\} & (p \geq 2,\ l \neq 0,\ \varepsilon = \operatorname{sgn} l), \\ \{V_0(\lambda, l) : \lambda > 0\} & (p \geq 2,\ l = 0), \\ \{V_\varepsilon(\lambda, l) : |l| - q \geq \lambda > 0\} & (p = 1,\ |l| > q,\ \varepsilon = \operatorname{sgn} l), \\ \emptyset & (p = 1,\ |l| \leq q), \end{cases}$$

Here, χ_l is a character of $U(1)$ and λ runs over $\lambda \in 2\boldsymbol{Z} + l + p + q + 1$ (resp. $\lambda \in 2\boldsymbol{Z} + l + q$).

Third, let $G'' = Sp(p, q) \supset K'' = Sp(p) \times Sp(q)$, and represent the root system of $\mathfrak{k}''$ as $\Delta(\mathfrak{k}'', \mathfrak{t}''^c) = \{\pm(h_i - h_j), \pm 2h_l : 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq p+q,\ 1 \leq l \leq p+q\}$. We define

I) For $p \geq 1$, $q \geq 1$, $\mathfrak{q}''_+ := \mathfrak{q}(2h_1 + h_{p+1})$, $L''_+ \simeq \boldsymbol{T}^2 \times Sp(p-1, q-1)$.

II) For $p \geq 2$, $q \geq 0$, $\mathfrak{q}''_0 := \mathfrak{q}(2h_1 + h_2)$, $L''_0 \simeq \boldsymbol{T}^2 \times Sp(p-2, q)$.

For $\lambda \in \boldsymbol{N}_+$, $j \in \boldsymbol{N}$ such that $j \equiv \lambda + 1 \mod 2$ we define $(\mathfrak{g}'', K'')$-modules by:

$$W_+(\lambda, j) \equiv W_+^{Sp(p,q)}(\lambda, j) := (\mathcal{R}_{\mathfrak{q}''_+}^{\mathfrak{g}''})^{2p+2q-2}(\boldsymbol{C}_{\frac{\lambda+j+1}{2}h_1 + \frac{-\lambda+j+1}{2}h_{p+1}}) \text{ if } j+1 > \lambda,\ pq \geq 1,$$

$$W_0(\lambda, j) \equiv W_0^{Sp(p,q)}(\lambda, j) := (\mathcal{R}_{\mathfrak{q}''_0}^{\mathfrak{g}''})^{4p-4}(\boldsymbol{C}_{\frac{\lambda+j+1}{2}h_1 - \frac{-\lambda+j+1}{2}h_2}) \text{ if } \lambda \geq j+1,\ p \geq 2.$$

If $q \geq 1$, then we have (cf. [6] Theorem 1):

(3.2)(a) $W_+(\lambda, j)$, $W_0(\lambda, j)$ are non-zero and irreducible $(\mathfrak{g}'', K'')$-modules with $Z(\mathfrak{g}'')$-infinitesimal character $\left(\frac{\lambda+j+1}{2}, \frac{-\lambda+j+1}{2}, Q'', Q''-1, \ldots, 1\right)$ in the Harish-Chandra parametrization, where $Q'' := p+q-2$.

(3.2)(b) $\mathrm{Disc}(Sp(p, q)/Sp(1) \times Sp(p-1, q); \sigma_j)$ $(q \geq 1$ and $j \in \boldsymbol{N})$ are given by,

$$\begin{cases} \{W_0(\lambda, j) : \lambda > j\} \cup \{W_+(\lambda, j) : j > \lambda > 0\} & (p \geq 2), \\ \{W_+(\lambda, j) : j - 2q + 1 \geq \lambda > 0\} & (p = 1, j \geq 2q), \\ \emptyset & (p = 1, j < 2q). \end{cases}$$

Here, σ_j is the irreducible $j+1$ dimensional representation of $Sp(1)$. In (3.2)(b), λ runs over $\lambda \in 2\boldsymbol{Z} + j + 1$ and the multiplicity of discrete series is uniformly $j+1$ or 0.

We write $\mathscr{H}^k(\boldsymbol{R}^p)$ for spherical harmonics on S^{p-1} of degree $k (k \in \boldsymbol{N})$, which is isomorphic to $U_+^{SO(p)}\left(k + \frac{1}{2}p - 1\right)$ if $p \geq 3$ or $(p, k) = (2,0)$, to $U_+^{SO(p)}(k) \oplus U_-^{SO(p)}(k)$ if $p = 2$ and $k \geq 1$. If $p = 1$, we put $\mathscr{H}^k(\boldsymbol{R}^1) := \boldsymbol{C}$ for $k = 0, 1$ and $:= 0$ for $k \geq 2$. Next, we write spherical harmonics of degree $(\alpha, \beta)(\alpha, \beta \in \boldsymbol{N})$ as $\mathscr{H}^{\alpha,\beta}(\boldsymbol{C}^p) = V_0^{U(p)}(\alpha+\beta+p-1, \alpha-\beta) \subset \mathscr{H}^{\alpha+\beta}(\boldsymbol{R}^{2p})$ $(p \geq 2)$. In the case $p = 1$, it is non-zero only if $\alpha\beta = 0$. Finally, we write $F^{Sp(p)}(x, y)$ $(x \geq y \geq 0)$ for the irreducible representation of $Sp(p)$ with an extremal weight $xf_1 + yf_2$. In the case $p = 1$, it is non-zero only if $y = 0$.

Theorem 3.3 $(SO_0(p, q) \downarrow SO_0(p, s) \times SO(q-s))$. *Let* $p \geq 2$, $s \geq 1$, $q - s \geq 1$, $\lambda \in \boldsymbol{Z} + \frac{1}{2}(p+q)$, $\lambda > 0$.

$$U_+^{SO_0(p,q)}(\lambda)_{|SO_0(p,s)\times SO(q-s)} \simeq \bigoplus_{a,k\in\boldsymbol{N}} U_+^{SO_0(p,s)}\left(\lambda + \frac{1}{2}(q-s) + a + 2k\right) \boxtimes \mathscr{H}^a(\boldsymbol{R}^{q-s}).$$

Theorem 3.4 $(U(p, q) \downarrow U(p, s) \times U(q-s))$. *Let* $s \geq 1$, $q - s \geq 1$, $\lambda \in \boldsymbol{N}_+$, $l \in 2\boldsymbol{Z} + \lambda + p + q + 1$. *For convenience, we define an irreducible representation of* $U(p, s) \times U(q-s)$ *by*

$$\mathscr{V}_\delta(\alpha, \beta, k; \lambda, l) := V_\delta^{U(p,q)}(\lambda + q - s + \alpha + \beta + 2k, l - \alpha + \beta) \boxtimes \mathscr{H}^{\alpha,\beta}(\boldsymbol{C}^{q-s}).$$

1)(i) *Suppose* $p \geq 2$, $l > \lambda + q - s$.

$$V_+^{U(p,q)}(\lambda, l)_{|U(p,s)\times U(q-s)} \simeq \bigoplus_{\substack{\alpha,\beta,k\in\boldsymbol{N} \\ \alpha+k<\frac{1}{2}(l-\lambda-q+s)}} \mathscr{V}_+(\alpha, \beta, k; \lambda, l) \oplus \bigoplus_{\substack{\alpha,\beta,k\in\boldsymbol{N} \\ \alpha+k\geq\frac{1}{2}(l-\lambda-q+s)}} \mathscr{V}_0(\alpha, \beta, k; \lambda, l).$$

(ii) *Suppose* $p \geq 2$. *We put* $\delta = +$ *if* $\lambda + q - s \geq l > \lambda$ *and* $\delta = 0$ *if* $\lambda \geq l \geq -\lambda$ *in the left side.*

$$V_\delta^{U(p,q)}(\lambda, l)_{|U(p,s)\times U(q-s)} \simeq \bigoplus_{\alpha,\beta,k\in\boldsymbol{N}} \mathscr{V}_0(\alpha, \beta, k; \lambda, l).$$

Use the duality (3.1)(*b*) *if* $-\lambda > l \geq -\lambda - q + s$ *or* $-\lambda - q + s > l$.

2) *Suppose* $p = 1$, $l \geq \lambda + q$. (*Use the duality* (3.1)(*b*) *if* $-l \geq \lambda + q$.)

$$V_+^{U(1,q)}(\lambda, l)_{|U(1,s)\times U(q-s)} \simeq \bigoplus_{\substack{\alpha,\beta,k\in\boldsymbol{N} \\ \alpha+k\leq\frac{1}{2}(l-\lambda-q)}} \mathscr{V}_+(\alpha, \beta, k; \lambda, l).$$

Theorem 3.5 ($Sp(p, q) \downarrow Sp(p, s) \times Sp(q-s)$). *Let* $s \geq 1, q-s \geq 1$, $\lambda \in \boldsymbol{N}_+, j \in \boldsymbol{N}, j \in 2\boldsymbol{Z}+\lambda+1$. *For convenience, we define an irreducible representation of* $Sp(p, s) \times Sp(q-s)$ *by* $\mathcal{W}_\delta(y, v, k, t ; \lambda, j) :=$
$W_\delta^{Sp(p,s)}(\lambda+2q-2s+2y+2k+t+v, j+v-t) \boxtimes F^{Sp(q-s)}(y+t+v, y)$.

1)(i) *Suppose* $p \geq 2, j+1 > \lambda+2q-2s$. Then, $W_+^{Sp(p,q)}(\lambda, j)_{|Sp(p,s)\times Sp(q-s)} \simeq$

$$\bigoplus_{\substack{y,v,k,t \in \boldsymbol{N} \\ 0 \leq t \leq j \\ y+k+t<\frac{1}{2}(j+1-\lambda)-q+s}} \mathcal{W}_+(y, v, k, t ; \lambda, j) \oplus \bigoplus_{\substack{y,v,k,t \in \boldsymbol{N} \\ 0 \leq t \leq j \\ y+k+t \geq \frac{1}{2}(j+1-\lambda)-q+s}} \mathcal{W}_0(y, v, k, t ; \lambda, j).$$

(ii) *Suppose* $p \geq 2$. *We put* $\delta = +$ *if* $\lambda+2q-2s \geq j+1 > \lambda$ *and* $\delta = 0$ *if* $p \geq 2$, $\lambda \geq j+1$ *in the left side.*

$$W_\delta^{Sp(p,q)}(\lambda, j)_{|Sp(p,s)\times Sp(q-s)} \simeq \bigoplus_{\substack{y,v,k,t \in \boldsymbol{N} \\ 0 \leq t \leq j}} \mathcal{W}_0(y, v, k, t ; \lambda, j).$$

2) *Suppose* $p=1, j \geq \lambda+2q-1$.

$$W_+^{Sp(p,q)}(\lambda, j)_{|Sp(1,s)\times Sp(q-s)} \simeq \bigoplus_{\substack{y,v,k,t \in \boldsymbol{N} \\ y+k+t \leq \frac{1}{2}(j+1-\lambda)-q}} \mathcal{W}_+(y, v, k, t ; \lambda, j).$$

A detailed proof is to appear elsewhere.

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