# 55. *A Remark on the Limiting Absorption Method for Dirac Operators**)

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1. **Introduction and result.** Let us consider the Dirac operator

$$H = \sum_{j=1}^{3} \alpha_j D_j + \beta + q(x),\ x \in \boldsymbol{R}^3,\ D_j = -i\frac{\partial}{\partial x_j},$$

in the Hilbert space $[L^2(\boldsymbol{R}^3)]^4$, where $\alpha_j$ and $\alpha_4 = \beta$ are $4 \times 4$ Hermitian constant matrices satisfying the anti-commutation property

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I \quad (1 \leq j,\ k \leq 4),$$

and $q(x)$ is a continuous real valued function which decays at infinity, where $I$ is the unit $4 \times 4$ matrix. For a real number $t$, let $L^2_t(\boldsymbol{R}^N)$ be the weighted Hilbert space with the norm

$$\| f \|_t = \left\{ \int_{\boldsymbol{R}^N} (1 + |x|^2)^t \, |f(x)|^2 \, dx \right\}^{1/2} < \infty$$

and let $X_t$ be a the weighted Hilbert space defined by $[L^2_t(\boldsymbol{R}^3)]^4$ (,where we use also the same notation $\| \ \|_t$ as the norm). One can see by the limiting absorption method under appropriate conditions on $q(x)$ that

*for any* $t > \frac{1}{2}$ *and any* $f \in X_t$ *the strong limit of the resolvent*

$$R(\lambda \pm i0)f = \text{s} - \lim_{\varepsilon \to +0} (H - \lambda \mp i\varepsilon)^{-1} f \quad \text{in } X_{-t}$$

*exists for any real* $\lambda$ *such that* $|\lambda| > 1$ (see, e. g., Yamada [6]).

For Schrödinger operators $h = -\Delta + q(x)$ in $\boldsymbol{R}^N$ there are also many works on the limiting absorption method, which shows that

*for any* $t > \frac{1}{2}$ *and any* $f \in L^2_t(\boldsymbol{R}^N)$ *the strong limit of the resolvent*

$$r(\lambda \pm i0)f = \text{s-}\lim_{\varepsilon \to +0} (h - \lambda \mp i\varepsilon)^{-1} f \quad \text{in } L^2_{-t}(\boldsymbol{R}^N)$$

*exists for any* $\lambda > 0$.

Let us dentote the operator norm of $r(\lambda \pm i0)$ ($R(\lambda \pm i0)$) as a bounded operator on $L^2_t$ to $L^2_{-t}$ (on $X_t$ to $X_{-t}$) by $\| r(\lambda \pm i0) \|_{t,-t}$ ($\| R(\lambda \pm i0) \|_{t,-t}$).

It is well known that the operator norm $\| r(\lambda \pm i0) \|_{t,-t}$ for each $t > \frac{1}{2}$ satisfies

$$\| r(\lambda \pm i0) \|_{t,-t} = O(\lambda^{-1/2}) \text{ as } \lambda \to \infty$$

for Schrödinger operators with a large class of potentials (see, e.g., Saitō [3],

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[4]). This property is one of important tools in the spectral and scattering theory of Schrödinger equations (see, e.g., Ben-Artzi [1], Saito [5]).

Our aim in this paper is to give an answer to Prof. Y. Saitō's problem of Dirac operators;

*whether the operator norm* $\| R(\lambda \pm i\,0) \|_{t,-t}$ *decays as* $|\lambda| \to \infty$ *or not.*

Our result is the following

**Proposition 1.** *Let us denote* $H$ *and* $R(\lambda \pm i\,0)$ *by* $H_0$ *and* $R_0(\lambda \pm i\,0)$, *if* $q(x) \equiv 0$. *Then, the operator norm* $\| R_0(\lambda \pm i\,0) \|_{t,-t}$ *does not decay as* $|\lambda| \to \infty$ *for any* $t > \frac{1}{2}$.

The above proposition is a direct result of the following lemma, which will be proved in the next section.

**Lemma 2.** *There exists a bounded sequence* $\{f_n\}$ *in* $X_t$ *for any* $t > \frac{1}{2}$ *such that*

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^3} \langle (R_0(n \pm i\,0) f_n)(x), f_n(x) \rangle \, dx \neq 0, \tag{1}$$

*where* $\langle \ , \ \rangle$ *denotes the usual inner product in* $\boldsymbol{C}^4$.

We remark here that the similar sequence to $\{f_n\}$ in Lemma 2 can be constructed, when "$n \to \infty$" is replaced by "$n \to -\infty$".

*Proof of Proposition* 1. Assume that

$$\| R_0(\lambda \pm i\,0) \|_{t,-t} \to 0 \text{ as } |\lambda| \to \infty$$

for some $t > \frac{1}{2}$. Then we take such a sequence $\{f_n\}$ as in Lemma 2, satisfying $\| f_n \|_t \leq C$ for some positive constant $C$ independent of $n$. Then we have

$$\left| \int_{\boldsymbol{R}^3} \langle (R_0(n \pm i\,0) f_n)(x), f_n(x) \rangle \, dx \right| \leq \| R_0(n \pm i\,0) f_n \|_{-t} \| f_n \|_t$$
$$\leq \| R_0(n \pm i\,0) \|_{t,-t} \| f_n \|_t^2 \leq C^2 \| R_0(n \pm i\,0) \|_{t,-t} \to 0, \text{ as } n \to \infty,$$

which is a contradiction to (1). Q. E. D.

**2. Proof of Lemma 2.** In this section we construct the sequence $\{f_n\}$ in Lemma 2.

The matrix $\beta$ has the eigenvalue 1 in view of the anti-commutation property of $\alpha_j$ and $\beta$. Let $g$ be a unit eigenvector of the matrix $\beta$ corresponding to the eigenvalue 1 and $\rho(s)$ be a real valued even function on $\boldsymbol{R}$ such that $\rho \in C^\infty$ and

$$\rho(s) \geq 0, \ \rho(0) = 1, \quad \operatorname{supp}[\rho] = [-1, 1].$$

and put

$$\varphi_n(\xi) = \frac{\rho(-n + \sqrt{1 + |\xi|^2})}{|\xi|}.$$

Then we define $f_n(x)$ by the inverse Fourier transform of $\varphi_n(\xi) g$ ($n = 2, 3, \cdots$), i.e.,

$$\varphi_n(\xi) g = \hat{f}_n(\xi) = (2\pi)^{-3/2} \int_{\boldsymbol{R}^3} e^{-ix\xi} f_n(x) \, dx$$

which are in $[C_0^\infty(\boldsymbol{R}_\xi^3)]^4$. Simple calculation yields

$$\|\varphi_n\|_0^2 = 4\pi\int_0^\infty \rho(-n+\sqrt{1+r^2})^2\,dr$$

$$= 4\pi\int_1^\infty \rho(s-n)^2\frac{s}{\sqrt{s^2-1}}\,ds \le \text{const.} \quad (n=2,3,\cdots)$$

and

$$\left\|\frac{\partial}{\partial|\xi|}\varphi_n(\xi)\right\|_0 = \left\| -\frac{\rho(-n+\sqrt{1+|\xi|^2})}{|\xi|^2} + \frac{\rho'(-n+\sqrt{1+|\xi|^2})}{\sqrt{1+|\xi|^2}}\right\|_0$$
$$\le \text{const.} \quad (n=2,3,\cdots).$$

In the same way we obtain

$$\left\|\left(\frac{\partial}{\partial|\xi|}\right)^k\varphi_n(\xi)\right\|_0 \le \text{const.} \quad (n=2,3,\cdots),$$

and

$$\|(1-\Delta_\xi)^k\varphi_n(\xi)\|_0 \le \text{const.} \quad (n=2,3,\cdots)$$

for each integer $k$. Thus, the sequence $\{f_n\}$ is a bounded sequence in $X_t$ for each $t>0$.

Now let us prove (1). Let

$$\Psi_\pm(\xi) = \frac{1}{2}\left(I \pm \frac{\sum_{j=1}^3 \xi_j\alpha_j + \beta}{\sqrt{1+|\xi|^2}}\right).$$

Then it follows (from Lemma 3.10 in [7]) that

$$(R_0(z)f_n, f_n)$$
$$= \int_{\boldsymbol{R}^3}\Big\{\frac{1}{\sqrt{1+|\xi|^2}-z}\langle\Psi_+(\xi)\hat{f}_n(\xi), \hat{f}_n(\xi)\rangle - \frac{1}{\sqrt{1+|\xi|^2}+z}\langle\Psi_-(\xi)\hat{f}_n(\xi), \hat{f}_n(\xi)\rangle\Big\}\,d\xi$$

for any non-real $z$, where ( , ) denotes the inner product in $X_0 = [L^2(\boldsymbol{R}^3)]^4$. Noticing that $\hat{f}_n(\xi)$ are functions of $|\xi|$ only and

$$\beta g = g,\ |g| = 1,\ \int_{|\xi|=1}\Big(\sum_{j=1}^3 \xi_j\alpha_j\Big)\,dS = 0,$$

we have

$$(R_0(z)f_n, f_n)$$
$$(2)\quad = 2\pi\int_0^\infty \Big\{\frac{1}{\sqrt{1+r^2}-z}\Big(1+\frac{1}{\sqrt{1+r^2}}\Big) - \frac{1}{\sqrt{1+r^2}+z}\Big(1-\frac{1}{\sqrt{1+r^2}}\Big)\Big\}|\varphi_n(r)|^2 r^2\,dr$$
$$= 2\pi\int_1^\infty \Big\{\frac{1}{s-z}\Big(1+\frac{1}{s}\Big) - \frac{1}{s+z}\Big(1-\frac{1}{s}\Big)\Big\}\rho(s-n)^2\frac{s}{\sqrt{s^2-1}}\,ds.$$

Making $z\to n\pm i\,0$ in (2), we obtain by means of Privalov's theorem on Cauchy's integral

$$\int_{\boldsymbol{R}^3}\langle(R_0(n\pm i\,0)f_n)(x), f_n(x)\rangle\,dx = \pm 2\pi^2 i\Big(1+\frac{1}{n}\Big)\frac{n}{\sqrt{n^2-1}}\rho(0)^2$$

$$(3) \qquad + 2\pi \text{ p.v.} \int_{-1}^{1} \frac{1}{s}\left(1 + \frac{1}{s+n}\right)\rho(s)^2 \frac{s+n}{\sqrt{(s+n)^2-1}}\,ds$$
$$- 2\pi \int_{-1}^{1} \frac{1}{s+2n}\left(1 - \frac{1}{s+n}\right)\rho(s)^2 \frac{s+n}{\sqrt{(s+n)^2-1}}\,ds,$$

where "p.v." means the *principal value*. Putting

$$w(s) = \left(1 + \frac{1}{s}\right)\frac{s}{\sqrt{s^2-1}},$$

we have

$$(3) = \pm\, 2\pi^2 i\, w(n)\, \rho(0)^2 + 2\pi\, w(n) \text{ p.v.} \int_{-1}^{1} \frac{\rho(s)^2}{s}\,ds$$
$$(4) \qquad + 2\pi \int_{-1}^{1} \frac{1}{s}\{w(s+n) - w(n)\}\rho(s)^2\,ds$$
$$- 2\pi \int_{-1}^{1} \frac{1}{s+2n}\left(1 - \frac{1}{s+n}\right)\frac{s+n}{\sqrt{(s+n)^2-1}}\,ds.$$

Noting that

$$w(n) \to 1, \text{ as } n \to \infty,$$
$$\sup\left\{\left|\frac{w(s+n) - w(n)}{s}\right|; 0 < |s| \le 1\right\} \to 0, \text{ as } n \to \infty,$$

and letting $n \to \infty$ in (4), we have

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^3} \langle (R_0(n \pm i\,0) f_n)(x), f_n(x)\rangle\,dx$$
$$= \pm\, 2\pi^2 i\, \rho(0)^2 + 2\pi \text{ p.v.} \int_{-1}^{1} \frac{\rho(s)^2}{s}\,ds = \pm\, 2\pi^2 i,$$

since $\rho(s)$ is an even function on $\boldsymbol{R}$ satisfying $\rho(0) = 1$. Q. E. D

Finally, we note that there is a recent work by C. Pladdy, Y. Saitō and T. Umeda on the asymptoic behavior of the resolvent of Dirac operators (C. Pladdy, Y. Saitō and T. Umeda [2]).

## References


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