54. *Gauss Decomposition of Connection Matrices and Application to Yang-Baxter Equation. I*

By Kazuhiko AOMOTO*) and Yoshifumi KATO**)

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1. General scheme. Method of Gauss decomposition. Suppose that an $N \times N$ matrix $G = ((g_{i,j}))_{i,j=1}^{N}$ is given such that all the entries $g_{i,j}$ are functions of $x = (x_1, \dots, x_m) \in (\boldsymbol{C}^*)^m$. Let $\mathfrak{S}_m$ be the symmetric group of m th degree with the canonical generators $\tau_1, \dots, \tau_{m-1}$, τ_j being the tranposition between the arguments x_j and x_{j+1}. Then we have the Coxeter relations, $\tau_j^2 = e$, $\tau_j \tau_{j+1} \tau_j = \tau_{j+1} \tau_j \tau_{j+1}$ (e denotes the identity). Let $S : \mathfrak{S}_m \ni \tau \to S_\tau \in GL_N(\boldsymbol{C})$ be a linear representation of $\mathfrak{S}_m$, i.e., $S_{\tau\tau'} = S_\tau S_{\tau'}$ and $S_e = 1$ and $\tau, \tau' \in \mathfrak{S}_m$.

We firstly assume the following property for $G(x)$.

(1.1) $$\tau G(x) \ \text{(defined as } G(\tau^{-1}(x))) \mp S_\tau^{-1} \cdot G(x) \cdot S_\tau$$

for an arbitrary $\tau \in \mathfrak{S}_m$.

Let $G(x) = \Omega^*(x)^{-1} \cdot \Omega(x)$ be a Gauss decomposition of $G(x)$ such that $\Omega(x)$ is a lower triangular matrix and $\Omega^*(x)$ is an upper triangular one. Let the one cocycle $\{W_\tau(x)\}_{\tau \in \mathfrak{S}_m}$ with values in $GL_N(\boldsymbol{C})$ be defined as $W_\tau(x) = \Omega(x) \cdot S_\tau \cdot (\tau\Omega(x))^{-1} = \Omega^*(x) \cdot S_\tau \cdot (\tau\Omega^*(x))^{-1}$ so that we have

(1.2) $$W_{\tau\tau'}(x) = W_\tau(x) \cdot \tau W_{\tau'}(x) \text{ and } W_e(x) = 1.$$

Secondly we assume that each $W_{\tau_r}(x)$ depends only on x_{r+1}/x_r, i.e.,

(1.3) $$W_{\tau_r}(x) = W_r(x_{r+1}/x_r), \quad 1 \le r \le m-1.$$

The equalities $W_{\tau_j \tau_{j+1} \tau_j} = W_{\tau_{j+1} \tau_j \tau_{j+1}}$ and $W_{\tau_j^2} = 1$ and the cocycle condition shows the Yang-Baxter equation

(1.4) $$W_j(u) W_{j+1}(uv) W_j(v) = W_{j+1}(v) W_j(uv) W_{j+1}(u)$$

and

(1.5) $$W_j(u)\, W_j(u^{-1}) = 1.$$

Then we call the matrix $G(x)$ *admissible*. Admissible matrices appear in a natural manner as connection coefficients among the symmetric A-type Jackson integrals. In this note we shall state their explicit formulae without proof.

2. Symmetric A-type Jackson integrals and the associated principal connection matrices. Let $q \in \boldsymbol{C}$, $|q| < 1$, be the elliptic modulus. We consider the symmetric A-type q-multiplicative function $\Phi_{n,m}(t)$ of $t = (t_1, \dots, t_n)$ on the n-dimensional algebraic torus $(\boldsymbol{C}^*)^n$,

(2.1) $$\Phi_{n,m}(t) = \Phi_{n,m}(t \mid x ; \alpha_1, \beta, \gamma)$$
$$= \Pi_{j=1}^{n} \left\{ t_j^{\alpha_1} \Pi_{k=1}^{m} \frac{(t_j/x_k)_\infty}{(t_j q^\beta / x_k)_\infty} \right\} \Pi_{1 \le i < j \le n} \frac{(q^{\gamma'} t_j / t_i)_\infty}{(q^\gamma t_j / t_i)_\infty}$$

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*) Department of Mathematics, Nagoya University.

**) Department of Mathematics, Meijo University.

for complex numbers $\alpha_1, \beta, \gamma, \gamma'$ and $x=(x_1,\dots,x_n)\in(\boldsymbol{C}^*)^m$ such that $\alpha_j=\alpha_1+(j-1)(\gamma'-\gamma)$ and $\gamma+\gamma'=1$. $\Phi_{n,m}(t)$ is a quasi-symmetric function of t in the following sense,

(2.2) $\sigma\Phi_{n,m}(t\mid x;\alpha_1,\beta,\gamma)$ (defined as $\Phi_{n,m}(\sigma^{-1}(t)\mid x;\alpha_1,\beta,\gamma)$)
$= U_\sigma(t)\,\Phi_{n,m}(t\mid x;\alpha_1,\beta,\gamma)$, for $\sigma\in\mathfrak{S}_m$

where $U_\sigma(t)$ denotes the pseudo-constant given by

$$(2.3)\qquad U_\sigma(t)=\Pi_{\substack{1\le i<j\le n\\ \sigma^{-1}(i)>\sigma^{-1}(j)}}\left(\frac{t_j}{t_i}\right)^{\gamma-\gamma'}\frac{\theta(q^\gamma t_j/t_i)}{\theta(q^{\gamma'}t_j/t_i)}$$

for the Jacobi elliptic theta function $\theta(u)=(u)_\infty(q/u)_\infty(q)_\infty$, $(u)_\infty=\Pi_{\nu=1}^\infty(1-uq^\nu)$. $\{U_\sigma(t)\}_{\sigma\in\mathfrak{S}_n}$ is a one-cocycle defined on $(\boldsymbol{C}^*)^n$ with values in $\boldsymbol{C}^*$.

We put $\Phi_{n,m}^{(a)}(t)=\Phi_{n,m}(t)D(t)$, $\Phi_{n,m}(t)$ being multiplied by the difference product $D(t)=\Pi_{1\le i<j\le n}(t_i-t_j)$. We are interested in Jackson integrals of the quasi skew symmetric q-multiplicative function $\Phi_{n,m}^{(a)}(t)$. These are essentially the same as the ones investigated in [10], [12], [14] appearing in correlation functions of quantum analogues of conformal field theory. These are also related with representation theory of quantum affine algebras. However we do not discuss it here.

Definition 1. The Jackson integral of $\Phi_{n,m}^{(a)}(t)$ over the cycle $[0,\xi_\infty]_q$ is, by definition, equal to the sum over the lattice in $(\boldsymbol{C}^*)^n$,

$$(2.4)\ \int_{[0,\xi\infty]_q}\Phi_{n,m}^{(a)}(t\mid x;\alpha_1,\beta,\gamma)\,\tilde{\omega}=\Sigma_{\chi\in\boldsymbol{Z}^n}(1-q)^n\,\Phi_{n,m}^{(a)}(\xi q^\chi\mid x;\alpha_1,\beta,\gamma)$$

for some $\xi\in(\boldsymbol{C}^*)^n$, where $\tilde{\omega}$ denotes the q-logarithmic n-form $\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_n}{t_n}$. When this sum has no meaning or diverges, a suitable regularization is necessary. For the moment we do not make it precise.

Let $F=\langle f_1,\dots,f_m\rangle$ be a partition of n such that $f_j\ge 0$, $\Sigma_{j=1}^m f_j=n$. The number of such partitions is equal to $\binom{n+m-1}{m-1}$. We want to define special cycles $[0,\xi\infty]_q$ or their regularization by specifying ξ depending on the partition F as follows.

Definition 2. We denote by $Y_{f_1,\dots,f_m}^+$ the set of points $(\boldsymbol{C}^*)^n$ such that $t_k=x_rq^{1+\nu_k+(k-f_1-\cdots-f_{r-1}-1)\gamma}$, $\nu_k\in\boldsymbol{Z}_{\ge0}$ for $f_1+\cdots+f_{r-1}+1\le k\le f_1+\cdots+f_r$, $1\le r\le m$. Similarly we denote by $Y_{f_1,\dots,f_m}^-$ the set of points $t\in(\boldsymbol{C}^*)^n$ such that $t_k=x_rq^{-\beta-\nu_k-(k-f_1-\cdots-f_{r-1}-1)\gamma}$, $\nu_k\in\boldsymbol{Z}_{\ge0}$. Since $\Phi_{n,m}^{(a)}(t)$ has poles at each point of $Y_{f_1,\dots,f_m}^-$, we take its residue there, i.e., the regularized cycle $\mathrm{reg}\,Y_{f_1,\dots,f_m}^-$ is defined such that we have

$$(2.5)\quad \int_{\mathrm{reg}\,Y_{f_1,\dots,f_m}^-}\Phi_{n,m}^{(a)}(t)\,\tilde{\omega}=\Sigma_{\zeta\in Y_{f_1,\dots,f_m}^-}\mathrm{Res}_{t=\zeta}\left\{\Phi_{n,m}^{(a)}(t)\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}\right\}$$

(refer to [1], [4], [5] for details).

We denote by $\xi=v_{f_1,\dots,f_m}^+$ and $\eta=v_{f_1,\dots,f_m}^-$ the points in $(\boldsymbol{C}^*)^n$ defined by $\xi_F=(\xi_1,\dots,\xi_n)$ and $\eta_F=(\eta_1,\dots,\eta_n)$ such that $\xi_k=x_rq^{1+(k-f_1-\cdots-f_{r-1}-1)}$, and $\eta_k=x_rq^{-\beta-(k-f_1-\cdots-f_{r-1}-1)\gamma}$ for $f_1+\cdots+f_{r-1}+1\le k\le f_1+\cdots+f_r$

respectively. We may remark that $Y^+_{f_1,\dots,f_m}$ and $Y^-_{f_1,\dots,f_m}$ are contained in $[0, \xi_F\infty]_q$ and $[0, \eta_F\infty]_q$ respectively. They contain the summits $v^+_{f_1,\dots,f_m}$, $v^-_{f_1,\dots,f_m}$ respectively. We have the asymptotic behaviours for the Jackson integrals over $Y^+_{f_1,\dots,f_m}$ and $\operatorname{reg} Y^-_{f_1,\dots,f_m}$ as follows.

$$(2.6)\qquad \int_{Y^+_{f_1,\dots,f_m}} \Phi^{(a)}_{n,m}(t)\,\tilde{\omega} \sim C_1 \Pi^n_{j=1}\xi_j^{\alpha_j} \quad \text{for} \quad \operatorname{Re}\alpha_1 \to +\infty,$$

$$(2.7)\qquad \int_{\operatorname{reg} Y^-_{f_1,\dots,f_m}} \Phi^{(a)}_{n,m}(t)\,\tilde{\omega} \sim C_2 \Pi^n_{j=1}\eta_j^{\alpha_j} \quad \text{for} \quad \operatorname{Re}\alpha_1 \to -\infty,$$

for non-zero constants C_1, C_2. We call the cycles $Y^+_{f_1,\dots,f_m}$ and $\operatorname{reg} Y^-_{f_1,\dots,f_m}$ *α-stable* and *α-unstable* respectively.

The following proposition has been proved in [1] and [6].

Proposition 1. (i) *For an arbitrary $\xi \in (\boldsymbol{C}^*)^n$, we have the relations in the n dimensional homology $H_n((\boldsymbol{C}^*)^n, \Phi^{(a)}_{n,m}, \partial_q)$* (see [5]) *among the cycles $[0, \xi\infty]_q$ and $Y^+_{f_1,\dots,f_m}$, $\operatorname{reg} Y^-_{f_1,\dots,f_m}$*, i.e.,

$$(2.8)\qquad \int_{[0,\xi\infty]_q} \Phi^{(a)}_{n,m}(t)\,\tilde{\omega} = \Sigma_{\langle f_1,\dots,f_m\rangle}([0, \xi\infty]_q : Y^+_{f_1,\dots,f_m})_{\Phi^{(a)}_{n,m}} \cdot \int_{Y^+_{f_1,\dots,f_m}} \Phi^{(a)}_{n,m}(t)\,\tilde{\omega}$$

$$(2.9)\qquad = \Sigma_{\langle f_1,\dots,f_m\rangle}([0, \xi\infty]_q : \operatorname{reg}Y^-_{f_1,\dots,f_m})_{\Phi^{(a)}_{n,m}} \cdot \int_{\operatorname{reg}Y^-_{f_1,\dots,f_m}} \Phi^{(a)}_{n,m}(t)\,\tilde{\omega}$$

for suitable pseudo-constants $([0, \xi\infty]_q : Y^+_{f_1,\dots,f_m})_{\Phi^{(a)}_{n,m}}$ and $([0, \xi\infty]_q : \operatorname{reg} Y^-_{f_1,\dots,f_m})_{\Phi^{(a)}_{n,m}}$ respectively.

(ii)(2.10) $([0, \xi\infty]_q : \operatorname{reg} Y^-_{f_1,\dots,f_m})_{\Phi^{(a)}_{n,m}} =$

$$\Sigma_{\sigma',\sigma''\in\mathfrak{S}_n} \phi(\sigma'^{-1}(\xi), \sigma''^{-1}(\eta))\cdot\operatorname{sgn}(\sigma'\sigma'')\cdot U^{-1}_{\sigma'}(\xi)\cdot U_{\sigma''}(\eta)$$

for $\eta = v^-_{f_1,\dots,f_m}$, where, $\phi(\xi, \eta)$ denotes the pseudo-constant

$$(2.11)\ \phi(\xi, \eta) = (1-q)^n(q)^{3n}_\infty \Pi^n_{j=1}\left(\frac{\xi_j}{\eta_j}\right)^{\alpha_j} \frac{\theta(q^{\alpha_j+\cdots+\alpha_n+1}\xi_j\eta_{j-1}/(\xi_{j-1}\eta_j))}{\theta(q^{\alpha_j+\cdots+\alpha_n+1})\,\theta(q\xi_j\eta_{j-1}/(\xi_{j-1}\eta_j))}$$

for $\xi_0 = \eta_0 = 1$. The RHS of (2.10) *will be denoted by $\tilde{\phi}(\xi, \eta)$.*

A sketch of proof of (2.10) can be found in [5]. Details will be published elsewhere ([6]).

Consider the $\binom{n+m-1}{m-1} \times \binom{n+m-1}{m-1}$ connection matrix $G = G(x \mid \alpha_1)$ with entries $g_{F,F'}$ for $F = \langle f_1, \dots, f_m\rangle$ and $F' = \langle f'_1, \dots, f'_m\rangle$

$$(2.12)\qquad g_{F,F'} = (Y^+_{f_1,\dots,f_m} : \operatorname{reg} Y^-_{f'_1,\dots,f'_m})_{\Phi^{(a)}_{n,m}}$$

such that $f_1 + \cdots + f_m = f'_1 + \cdots + f'_m = n$. This is equal to

$$(2.13)\qquad \tilde{\phi}(v^+_{f_1,\dots,f_m}, v^-_{f'_1,\dots,f'_m}) = \lim_{\xi\to v^+_{f_1,\dots,f_m}} \tilde{\phi}(\xi, v^-_{f'_1,\dots,f'_m})$$
$$= \lim_{\eta\to v^-_{f'_1,\dots,f'_m}} \tilde{\phi}(v^+_{f_1,\dots,f_m}, \eta).$$

We call G *the principal connection matrix associated with the Jackson integrals* (2.4).

3. Basic formula. *Case where $m = 1$ and $n \geq 1$.*

In this case there is only one α-stable cycle or α-unstable cycle, Y^+_n or reg Y_n^-. We may assume $x_1 = 1$. We want to evaluate the pseudo-constants $([0, \xi\infty]_q : \mathrm{reg}\, Y_n^-)_{\Phi_{n,1}^{(a)}}$ and $(Y_n^+ : \mathrm{reg}\, Y_n^-)_{\Phi_{n,1}^{(a)}}$. We put

$$(3.1) \qquad B_n^{(+)}(\alpha_1) = \int_{Y_n^+} \Phi_{n,1}^{(a)}(t)\, \tilde{\omega},\ B_n^{(-)}(\alpha_1) = \int_{\mathrm{reg}\, Y_n^-} \Phi_{n,1}^{(a)}(t)\, \tilde{\omega}.$$

The following two propositions have been proved in [2]. See also [8], [13].

Proposition 3 (Askey-Habsieger-Kadell). *$B_n^{(+)}(\alpha_1)$ and $B_n^{(-)}(\alpha_1)$ can be evaluated as follows.*

(3.2)

$$B_n^{(+)}(\alpha_1) = q^{A_n} \Pi_{j=1}^n \frac{\Gamma_q(\beta + 1 + (j-1)\gamma)\Gamma_q(\alpha_1 + n - 1 - (n+j-2)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)\Gamma_q(\alpha_1 + \beta + n - (n-j)\gamma)}$$

where $A_n = \sum_{j=1}^n (\alpha_j + n - j)(1 + (j-1)\gamma)$.

$$(3.3) \quad B_n^{(-)}(\alpha_1) = (-)^n (q)_\infty^{-n}\, q^{-C'_n}\, \Pi_{j=1}^n \frac{(q^{2-\alpha_1-n+(n+j-2)\gamma})_\infty (q^{-\beta-(j-1)\gamma})_\infty (q^{1-j\gamma})_\infty}{(q^{1-\alpha_1-\beta-n+(j-1)\gamma})_\infty (q^{1-\gamma})_\infty}$$

where $C'_n = \sum_{j=1}^n (\alpha_j + n - j)(\beta + (j-1)\gamma)$.

We can now state the

Basic formula. For an arbitrary $\xi \in (\boldsymbol{C}^*)^n$, we have

$$(3.4) \quad \tilde{\phi}(\xi, v_n^-) = \sum_{\sigma' \in \mathfrak{S}_n} \phi(\sigma'^{-1}(\xi), v_n^-)\ \mathrm{sgn}(\sigma')\ U_{\sigma'}^{-1}(\xi)$$

$$= (1-q)^n (q)_\infty^{3n}\, \Pi_{j=1}^n \left\{ \frac{(q^{(j-1)\gamma+\beta}\xi_j)^{\alpha_1 - 2(j-1)\gamma}\theta(q^{\alpha_1+\beta+1-(n-1)\gamma}\xi_j)}{\theta(q^{\alpha_1+1-(n+j-2)\gamma})\theta(q^{1+\beta}\xi_j)} \right\}.$$

$$(3.5) \qquad \Pi_{1 \le i < j \le n} \frac{\theta(q\xi_j/\xi_i)}{\theta(q^{1+\gamma}\xi_j/\xi_i)}.$$

This formula was stated in [2] and has been proved in [3].

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