51. On the Structure of Painlevé Transcendents with a Large Parameter[†)]

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§0. Introduction. The purpose of this note is to report a novel and intriguing result (Theorem 3.2 below) on the structure of Painlevé transcendents with a large parameter, which asserts that they can be locally reduced to a solution of the Painlevé I with a large parameter η (cf. Table 1.1 below). The details shall be published elsewhere.

The Painlevé equations with a large parameter to be discussed here naturally arise as conditions for isomonodromic deformations (in the sense of Jimbo and Miwa [3]) of certain Schrödinger equations (with a large parameter η) tabulated in §1. We hope the result reported here will turn out to be effective not only for the better understanding of the Painlevé transcendents but also for the computation of the monodromic structures of those equations in terms of WKB solutions (cf. [5], [2]). We sincerely thank Professors T. Aoki and M. Jimbo for the stimulating discussions on these topics, from which we have benefited much.

§1. List of Painlevé equations with a large parameter and associated Schrödinger equations. In order to fix our notations we list up Painlevé equations with a large parameter (Table 1.1) and the relevant Schrödinger equations (Table 1.2). The latter ones can be isomonodromically deformed if the unknown function $\psi_J (J = I, \dots, VI)$ satisfies the deformation equation

$$\frac{\partial \psi_J}{\partial t} = A_J(x, t, \lambda) \frac{\partial \psi_J}{\partial x} - \frac{1}{2} \frac{\partial A(x, t, \lambda)}{\partial x} \psi_J, \tag{1.1}$$

where A_J is the rational function tabulated in Okamoto [4], §4.4 (without the subscript J); for example,

$$A_J = \frac{1}{2(x-\lambda)} \quad (J = I, II), \tag{1.2}$$

$$A_{VI} = \frac{\lambda - t}{t(t-1)} \frac{x(x-1)}{x-\lambda}, \text{ etc.} \tag{1.3}$$

where λ is a solution of the Painlevé equation P_J tabulated below. Here and in what follows we use the symbol P_J to denote the J-th Painlevé equation with a large parameter η as specified below, although they differ from the original Painlevé equations in that they contain a large parameter. Similarly the symbol SL_J in Table 1.2 below denotes the Schrödinger equation with a large parameter. The parameter η is introduced into these equations in such

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a manner that they are compatible with the ordinary procedure of confluence of singularities in Painlevé equations (cf. e.g. [4]).

Table 1.1 (*Painlevé equations with a large parameter* η).

$$P_I : \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$P_{II} : \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha).$$

$$P_{III} : \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right].$$

$$P_{IV} : \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2 + 4\alpha_1)\lambda - \frac{4\alpha_0}{\lambda}\right].$$

$$P_V : \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda - 1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right)$$
$$+ \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0 + \alpha_\infty) - \alpha_0\frac{1}{\lambda^2} + \alpha_2\frac{t}{(\lambda-1)^2} - \alpha_1 t^2\frac{\lambda+1}{(\lambda-1)^3}\right].$$

$$P_{VI} : \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+ \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right.$$
$$\left. + \eta^2\left\{(\alpha_0 + \alpha_1 + \beta_1 + \alpha_\infty) - \alpha_0\frac{t}{\lambda^2} + \alpha_1\frac{t-1}{(\lambda-1)^2} - \beta_1\frac{t(t-1)}{(\lambda-t)^2}\right\}\right].$$

Definition 1.1. Let $F_J(\lambda, t)$ denote the coefficient of η^2 in P_J. We then denote by $F_J^\dagger(\lambda, t)$ the monic polynomial in λ that is obtained by multiplying F_J by a polynomial of λ and t.

In Table 1.2 below, we list up only the potential Q_J to specify the Schrödinger equation SL_J, i.e., $(-\partial^2/\partial x^2 + \eta^2 Q_J(x, t, \eta))\psi_J = 0$. The symbol K_J used there denotes the Hamiltonian given in [4], §4, that is, K_J is a t-dependent polynomial of λ and ν, with (λ, ν) obeying the Hamiltonian system

$$(1.4)_J \qquad \frac{d\lambda}{dt} = \eta\frac{\partial K_J}{\partial \nu}, \quad \frac{d\nu}{dt} = -\eta\frac{\partial K_J}{\partial \lambda}.$$

This system is known to be equivalent to the Painlevé equation P_J. (Cf. [4] and references cited there.) In particular,

$$(1.5) \qquad K_I = \nu^2/2 - 2\lambda^3 - t\lambda,$$

$$(1.6) \qquad K_{II} = \nu^2/2 - \lambda^4/2 - t\lambda^2/2 - \alpha\lambda, \qquad \text{etc.}$$

Note, however, that in this article λ and ν are, in addition to the constraint $(1.4)_J$, supposed to have the following formal series expansions in η^{-1}:

$$(1.7) \qquad \lambda = \lambda_0(t) + \lambda_1(t)\eta^{-1} + \lambda_2(t)\eta^{-2} + \cdots$$

$$(1.8) \qquad \nu = \nu_0(t) + \nu_1(t)\eta^{-1} + \nu_2(t)\eta^{-2} + \cdots.$$

Other constants such as α in (1.6) are supposed to be genuine costants, i.e., complex numbers. We also note that ν_0 will be seen to vanish (Proposition 2.1 (ii)).

Table 1.2 (*Potentials of SL_J with a large parameter η*).

$$Q_I = 4x^3 + 2tx + 2K_I - \frac{\eta^{-1}\nu}{x-\lambda} + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

$$Q_{II} = x^4 + tx^2 + 2\alpha x + 2K_{II} - \frac{\eta^{-1}\nu}{x-\lambda} + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

$$Q_{III} = \frac{\alpha_0 t^2}{x^4} + \frac{\alpha_0' t}{x^3} + \frac{\alpha_\infty' t}{x} + \alpha_\infty t^2 + \frac{tK_{III}}{2x^2} + \left(\frac{1}{2x^2} - \frac{1}{x(x-\lambda)}\right)\eta^{-1}\lambda\nu + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

$$Q_{IV} = \frac{\alpha_0}{x^2} + \alpha_1 + \left(\frac{x+2t}{4}\right)^2 + \frac{K_{IV}}{2x}^{(1)} - \frac{\eta^{-1}\lambda\nu}{x(x-\lambda)} + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

$$Q_V = \frac{\alpha_0}{x^2} + \frac{\alpha_1 t^2}{(x-1)^4} + \frac{\alpha_2 t}{(x-1)^3} + \frac{\alpha_\infty}{(x-1)^2} + \frac{tK_V}{x(x-1)^2} - \frac{\eta^{-1}\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)} + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

$$Q_{VI} = \frac{\alpha_0}{x^2} + \frac{\alpha_1}{(x-1)^2} + \frac{\alpha_\infty}{x(x-1)} + \frac{\beta_1}{(x-t)^2} + \frac{t(t-1)K_{VI}}{x(x-1)(x-t)} - \frac{\eta^{-1}\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)} + \frac{3\eta^{-2}}{4(x-\lambda)^2}.$$

§2. WKB analysis of SL_J and P_J. When a (non-zero) WKB solution ψ_J of the equation SL_J obeys the deformation equation (1.1), we observe several interesting phenomena concerning the structure of SL_J itself and the logarithmic derivative S_J of ψ_J, i.e., $S_J = \frac{\partial}{\partial x}\log\psi_J = S_{J,-1}\eta + S_{J,0} + S_{J,1}\eta^{-1} + \cdots$.(In what follows, if there is no fear of confusions, we will sometimes omit the subscript J in S_J.) We summarize them as Proposition 2.1 below. See [2] for the details.

Proposition 2.1. *Suppose that a non-zero WKB solution ψ_J of $SL_J(J = I, \cdots, VI)$ satisfies* (1.1), *and let $S = S_J$ denote its logarithmic derivative. Then we find the following*:

(i) *S_J satisfies the following equation*:

$$\frac{\partial S_J}{\partial t} = \frac{\partial}{\partial x}\left(A_J S_J - \frac{1}{2}\frac{\partial A_J}{\partial x}\right). \tag{2.1}$$

(ii) *The top term $\nu_0(t)$ in the expansion* (1.8) *vanishes identically, while the top term $\lambda_0(t)$ in* (1.7) *satisfies the following equation*:

(1) λ^2 in the definition of K_{IV} in [4], p. 615 is a misprinting of λ.

(2.2) $$F_J^\dagger(\lambda_0(t),\ t) = 0.$$

(iii) *Let $Q_{J,0}$ denote the part of Q_J that is homogeneous of degree* 0 *with respect to η. Then we find the following*:

(2.3) $$Q_{J,0}(x,\ t)\,|_{x=\lambda_0(t)} = \frac{\partial}{\partial x} Q_{J,0}(x,\ t)\,|_{x=\lambda_0(t)} = 0.$$

(iv) *$S_{J,j}(x,\ t)\,(j : odd)$ is holomorphic near $(x,\ t) = (\lambda_0(t),\ t)$, if t is not contained in $\Delta_J \underset{def}{=} \{t \in \boldsymbol{C}$; there exists λ such that $F_J^\dagger(\lambda,\ t) = \partial F_J^\dagger(\lambda,\ t)/\partial\lambda = 0\}$.*

Remark 2.1. The relation (2.1) entails that $\omega_J = S_J dx + \left(A_J S_J - \frac{1}{2}\frac{\partial A_J}{\partial x}\right)dt$ is a closed form. Hence using the WKB solution S_J of the Riccati equation with a parameter t, i.e., $S_J^2 + \partial S_J/\partial x = \eta^2 Q_J$, we can construct a WKB solution ψ_J of SL_J satisfying (1.1) by setting $\psi_J = \exp\left(\int^{(x,t)} \omega_J\right)$.

Starting from $\lambda_0(t)$ given in Proposition 2.1(ii), we can construct $\lambda_j(t)$ recursively by solving algebraic equations so that $\lambda_J = \sum_{j=0}^{\infty} \lambda_j(t)\eta^{-j}$ formally satisfies P_J. This formal series is pre-Borel-summable in the sense of [1]. Such a solution λ_J of P_J is our main concern in this article. Although we leave the detailed exact WKB analysis of λ_J to our subsequent articles, we introduce the following terminologies in the (exact) WKB analysis just to facilitate the description of our main result.

Definition 2.1. (i) A turning point for λ_J is, by definition, a point t which satisfies

(2.4) $$F_J^\dagger(\lambda_0(t),\ t) = \frac{\partial F_J^\dagger}{\partial\lambda}(\lambda_0(t),\ t) = 0.$$

Such a point t is said to be simple if

(2.5) $$\frac{\partial^2 F_J^\dagger}{\partial\lambda^2}(\lambda_0(t),\ t) \neq 0.$$

(ii) A Stokes curve for λ_J is the integral curve of the direction field $\mathrm{Im}\sqrt{\partial F_J(\lambda_0(t),\ t)/\partial\lambda}\,dt = 0$ that emanates from a turning point for λ_J.

The relevance of these notions to our current consideration is described in the following:

Proposition 2.2. (i) *For a simple turning point τ for λ_J, there exists a simple turning point $a(t)$ of SL_J, i.e., a simple zero $x = a(t)$ of $Q_{J,0}(x,\ t,\ \lambda_0(t))$, that merges with the (double) turning point $x = \lambda_0(t)$ at $t = \tau$.*

(ii) *In the situation described in* (i), *we find the following relation*:

(2.6) $$\int_{a(t)}^{\lambda_0(t)} \sqrt{Q_{J,0}(x,\ t,\ \lambda_0(t))}\,dx = \frac{1}{2}\int_{\tau}^{t} \sqrt{\frac{\partial F_J}{\partial\lambda}(\lambda_0(s),\ s)}\,ds.$$

The proof of (i) is straightforward, while the proof of (ii) makes essential use of (2.1). Considering the imaginary part of both sides in (2.6), we find that, if a point t lies in the Stokes curve for λ_J, there then exists a Stokes curve of SL_J that connects two turning points of SL_J, i.e., $\lambda_0(t)$ and $a(t)$.

§3. A local transformation between λ_J and λ_I. In this section we put ~

to variables and functions relevant to SL_J for the notational convenience, that is, symbols $\tilde{x}$, $\tilde{t}$, $\tilde{\lambda}_0(\tilde{t})$, etc. respectively designate the quantities x, t, $\lambda_0(t)$, etc. which appear in SL_J. Throughout this section, $\tilde{\sigma}$ denotes a point in a Stokes curve for $\tilde{\lambda}_J(\tilde{t})$ that emanates from a turning point $\tilde{\tau}$. Here we assume $\tilde{\sigma}$ is distinct from $\tilde{\tau}$. We denote by $\tilde{a}(\tilde{t})$ the simple turning point of SL_J that merges with $\tilde{\lambda}_0(\tilde{t})$ at $\tilde{t}=\tilde{\tau}$, whose existence is guaranteed by Proposition 2.2 (i). Note that such a simple turning point is unique in the case of SL_I, that is, $-2\lambda_0(t)$. We also denote by $\tilde{\gamma}$ the part of the Stokes curve that begins at $\tilde{a}(\tilde{t})$ and ends at $\tilde{\lambda}_0(\tilde{t})$; the existence of $\tilde{\gamma}$ is guaranteed by Proposition 2.2 (ii).

Theorem 3.1. *There exist a neighborhood $\tilde{V}$ of $\tilde{\sigma}$, a neighborhood $\tilde{U}$ of $\tilde{\gamma}$ and holomorphic functions $x_j(\tilde{x}, \tilde{t})$ $(j=0,1,2,\cdots)$ on $\tilde{U}\times\tilde{V}$ and $t_j(\tilde{t})$ $(j=0,1,2,\cdots)$ on $\tilde{V}$ so that the following relations may hold:*

(i) *The function $t_0(\tilde{t})$ satisfies*

$$\int_{\tilde{\tau}}^{\tilde{t}}\sqrt{\frac{\partial\tilde{F}_J}{\partial\tilde{\lambda}}}(\tilde{\lambda}_0(\tilde{s}),\tilde{s})\,d\tilde{s}=\int_0^t\sqrt{\frac{\partial F_I}{\partial\lambda}}(\lambda_0(s),s)\,ds\,\Big|_{t=t_0(\tilde{t})},\tag{3.1}$$

and, in particular, $dt_0/d\tilde{t}\neq 0$ holds on $\tilde{V}$ if $\tilde{V}$ is sufficiently small.

(ii) $x_0(\tilde{a}(\tilde{t}))=-2\lambda_0(t_0(\tilde{t}))$ *and* $x_0(\tilde{\lambda}_0(\tilde{t}))=\lambda_0(t_0(\tilde{t}))$.

(iii) $\partial x_0/\partial\tilde{x}\neq 0$ *on* $\tilde{U}\times\tilde{V}$.

(iv) *Letting $x(\tilde{x},\tilde{t},\eta)$ and $t(\tilde{t},\eta)$ respectively denote the formal series $\sum_{j\geq 0}x_j(\tilde{x},\tilde{t})\eta^{-j}$ and $\sum_{j\geq 0}t_j(\tilde{t})\eta^{-j}$, we find the following relation:*

$$\tilde{Q}_J(\tilde{x},\tilde{t},\eta)=\left(\frac{\partial x(\tilde{x},\tilde{t},\eta)}{\partial\tilde{x}}\right)^2 Q_I(x(\tilde{x},\tilde{t},\eta),t(\tilde{t},\eta),\eta)-\frac{1}{2}\eta^{-2}\{x(\tilde{x},\tilde{t},\eta);\tilde{x}\}.\tag{3.2}$$

Here $\{x;\tilde{x}\}$ denotes the Schwarzian derivative $\dfrac{\partial^3x/\partial\tilde{x}^3}{\partial x/\partial\tilde{x}}-\dfrac{3}{2}\left(\dfrac{\partial^2x/\partial\tilde{x}^2}{\partial x/\partial\tilde{x}}\right)^2$.

Remark 3.1. The functions x_j and t_j actually vanish identically for odd j's.

Remark 3.2. The roles of SL_J and SL_I are symmetric in the above result, that is, they can be interchanged.

Theorem 3.2. *Using the series $x(\tilde{x},\tilde{t},\eta)$ and $t(\tilde{t},\eta)$ in Theorem 3.1, we find the following:*

$$\lambda_I(t(\tilde{t},\eta),\eta)=x(\tilde{x},\tilde{t},\eta)\,|_{\tilde{x}=\tilde{\lambda}_J(\tilde{t},\eta)}.\tag{3.3}$$

The proof of Theorem 3.1 is similar to the proof of Theorem 3.1 in [1], where the "energy" E plays a role similar to the t-variable here. The proof of Theorem 3.2 makes full use of (2.1) and Proposition 2.1 (iv).

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