50. *Error Estimation of Newmark's Method for Conservative Second Order Linear Evolution Equation*

By Mihoko MATSUKI and Teruo USHIJIMA

Department of Computer Science and Information Mathematics,
The University of Electro-Communications

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1. Introduction. We consider the following continuous problem in a Hibert space X :

$$(E:\varphi^1,\varphi^0,\psi)\qquad \begin{cases} \dfrac{d^2}{dt^2}\varphi + A\varphi = \psi,\ t>0,\\ \varphi(0)=\varphi^1,\ \dfrac{d\varphi}{dt}(0)=\varphi^0 \end{cases}$$

with a positive definite selfadjoint operator A, and its fully discrete approximate problem in a given finite dimensional subspace X_h of X obtained by Newmark's method as follows:

$(E_{h,\tau}:\varphi^1_{h,\tau},\varphi^0_{h,\tau},\psi_{h,\tau,m})$

$$\begin{cases} \dfrac{\varphi_{h,\tau,m-1}-2\varphi_{h,\tau,m}+\varphi_{h,\tau,m+1}}{\tau^2}\\ \qquad + A_h\{(1-2\beta+\delta)\varphi_{h,\tau,m}+\beta\varphi_{h,\tau,m+1}+(\beta-\delta)\varphi_{h,\tau,m-1}\}\\ \qquad = \psi_{h,\tau,m},\qquad m=1,2,\cdots,\\ \varphi_{h,\tau,0}=\varphi^1_{h,\tau},\quad \dfrac{\varphi_{h,\tau,1}-\varphi_{h,\tau,0}}{\tau}=\varphi^0_{h,\tau} \end{cases}$$

with a bounded positive definite selfadjoint operator A_h, where β and δ are fixed nonnegative numbers independent of $h\in(0,\bar{h}]$, and τ is a positive number. For the derivation of the problem $(E_{h,\tau})$, we followed Raviart and Thomas [4].

Our motivation of this study is to analyze the finite element approximation of the linear water wave equation:

$$(\mathrm{LWW})\qquad \begin{cases} -\Delta\Phi = 0 & \text{in } \Omega,\ t>0,\\ \Phi_{tt}+g\dfrac{\partial\Phi}{\partial n} = F_t & \text{on } \Gamma_0,\ t>0,\\ \dfrac{\partial\Phi}{\partial n} = 0 & \text{on } \Gamma_1,\ t>0,\\ \Phi(0,x) = \Phi^1(x) & \text{on } \Gamma_0,\\ \Phi_t(0,x) = \Phi^0(x) & \text{on } \Gamma_0, \end{cases}$$

where $\partial\Omega=\overline{\Gamma_0\cup\Gamma_1}$ with mutually disjoint portions Γ_0 and Γ_1 of the boundary $\partial\Omega$ of the water region Ω at rest. The portion Γ_0 is the water surface at rest, and Γ_1 is the rigid wall. The problem (LWW) describes the motion of the water in a vessel on the ground of the Earth under the assumption of infinitesimal amplitude, and whose derivation from the fundamental laws of physics is given in Stoker [5], for example. The problem (LWW) can be represented as in the form (E) (see Ushijima and Matsuki [7]). In the problem the operator A is a realization of a first order pseudo-differential operator, which maps the boundary value on Γ_0 of a harmonic function in Ω to the value of the exterior normal derivative on Γ_0. The bounded operator A_h is a computable operator derived from the standard Galerkin approximation. It seems difficult to apply standard error estimation methods in the analysis of finite element approximation, such as the energy method and the semi-group theoretical method (see Fujita and Suzuki [1]). By this reason, we developed an abstract method to give the error estimation.

In the sequel, fix a complex Hilbert space X with inner product $(\, , \,)$ and norm $\| \ \|$. Let A be a given positive definite selfadjoint operator acting on the space X, and assume $A \geq \underline{\alpha}$ with a positive number $\underline{\alpha}$. Let $\bar{h}$ be an appropriate positive number. For any $h \in (0, \bar{h}]$, let X_h be a given finite dimensional subspace of X. Let P_h be the orthogonal projection from X onto X_h. For any $h \in (0, \bar{h}]$, let A_h be a given bounded positive definite selfadjoint operator acting on X_h. We assume that there exist positive numbers $\underline{\alpha}_h$ and $\bar{\alpha}_h$ with $\underline{\alpha} \leq \underline{\alpha}_h \leq A_h \leq \bar{\alpha}_h$.

The function $\varphi(t) \in X$, and the sequence $\{\varphi_{h,\tau,m} : m = 0,1,2, \cdots\} \subset X_h$, are unknowns which should be determined as solutions of the problem (E), and of $(E_{h,\tau})$, for given initial data $\varphi^i \in X (i = 0,1)$, and $\varphi^i_{h,\tau} \in X_h (i = 0,1)$, and given inhomogeneous data $\psi = \psi(t) \in X$, and $\{\psi_{h,\tau,m} : m = 1,2, \cdots\} \subset X_h$, respectively.

In this paper, we present the results of error estimates, and the strong convergence of the solution of fully discrete approximate problem, as Theorems 2 and 3, respectively. The details of the proof of them are reported in Matsuki and Ushijima [3].

2. Stability criterion. Define a bounded positive definite selfadjoint operator $A_{h,\tau}$ by $A_{h,\tau} = (1 + \beta\tau^2 A_h)^{-1} A_h$. We quote the following stability criterion from [7].

Theorem 1. *Fix $\gamma \in (0,1)$ independently of $h \in (0, \bar{h}]$. For any $h \in (0, \bar{h}]$, let τ be a positive number satisfying the following stability condition:*

$(S^{\delta}_{2\gamma})$
$$\tau^2 (1 + \delta)^2 \| A_{h,\tau} \| \leq (2\gamma)^2.$$

Then the solution $\{\varphi_{h,\tau,m} : m = 0,1,2, \cdots\}$ of $(E_{h,\tau})$ is estimated as follows.

$$\| \varphi_{h,\tau,m} \| \leq \left(1 + \frac{\gamma}{\sqrt{1-\gamma^2}}\right) \| \varphi^1_{h,\tau} \| + \frac{1}{\sqrt{1-\gamma^2}} \| A_{h,\tau}^{-\frac{1}{2}} \varphi^0_{h,\tau} \| + \frac{(m-1)\tau}{\sqrt{1-\gamma^2}} \cdot \max_{1 \leq j \leq m-1} \| \{(1 + \beta\tau^2 A_h) A_h\}^{-\frac{1}{2}} \psi_{h,\tau,j} \|, \quad m = 2,3, \cdots.$$

Proposition 1. *Fix $\gamma \in (0,1)$ independently of $h \in (0, \bar{h}]$. Define ρ as follows:*

$$(2.1) \qquad \rho = \begin{cases} \dfrac{2\gamma}{\{(1+\delta)^2 - \beta(2\gamma)^2\}^{1/2}}, & \text{if } \beta(2\gamma)^2 < (1+\delta)^2, \\ \text{arbitrary positive number independent of } h \in (0, \bar{h}], & \text{if } \beta(2\gamma)^2 \geq (1+\delta)^2. \end{cases}$$

Then the condition $(T_{2\gamma}^{\rho})$ $\quad \tau^2 \| A_h \| \le \rho^2 \quad$ *implies condition* $(S_{2\gamma}^{\delta})$.

Throughout this paper, let ρ be the positive number defined by (2.1).

3. Error estimates. Hereafter we assume the existence of a function $\varepsilon(h)$ defined on $(0, \bar{h}]$ satisfying the following conditions:

$(\varepsilon\text{-}0)$ $\quad \sup_{0<h\le\bar{h}} \varepsilon(h) = \bar{\varepsilon} < \infty,$

$(\varepsilon\text{-}1)$ $\quad \lim_{h\to 0} \varepsilon(h) = 0,$

$(\varepsilon\text{-}2)$ $\quad \left\{ \begin{array}{l} \text{There exists a positive number } \alpha \\ \text{independent of } h \in (0, \bar{h}] \\ \text{satisfying } \| A_h \| \le \dfrac{\alpha}{\varepsilon(h)}. \end{array} \right.$

Let s be a real number. We say that *the condition* $(A_{\varepsilon,s})$ *holds*, if there exists a constant C independent of h such that for any $h \in (0, \bar{h}]$ and $\varphi \in D(A^s)$ the following inequality satisfies:

$$\| (A_h^{-1} P_h - A^{-1})\varphi \| \le C\varepsilon(h)^{1+s} \| A^s \varphi \|.$$

We say that *the condition* (ε_s) *holds*, if the conditions $(A_{\varepsilon,s})$ and $(A_{\varepsilon,0})$ hold.

For the data φ^1, φ^0, ψ of continuous problem (E), we define the following condition (D_s) with real number s :

$$(D_s) \quad \left\{ \begin{array}{l} \varphi^i \in D(A^{s+\frac{i}{2}}),\ i = 0,1, \\ \psi \in C^2([0, \infty) : X),\ \psi(t),\ \psi^{(2)}(t) \in D(A^{s-1}),\ t \ge 0, \\ \psi^{(i)}(0) \in D(A^{s-\frac{1+i}{2}}),\ i = 0,1, \\ A^{s-1}\psi,\ A^{s-1}\psi^{(2)} \in C([0, \infty) : X). \end{array} \right.$$

Theorem 2. *Let s be a nonnegative number. Assume the conditions (D_s) and (ε_{s-1}). Fix $\gamma \in (0, 1)$ independently of $h \in (0, \bar{h}]$. Suppose that the positive number τ satisfies the condition $(S_{2\gamma}^{\delta})$, if $s \in [1, \infty)$, and the condition $(T_{2\gamma}^{\rho})$, if $s \in [0, 1]$. Suppose that $\varphi_{h,\tau}^1$, $\varphi_{h,\tau}^0 \in X_h$ and that $\{\varphi_{h,\tau,m}\,;\, m = 0,1,2,\cdots\}$ be an X_h-valued sequence. Then the difference between the generalized solution φ of continuous problem (E) at $t = m\tau$ and the solution $\varphi_{h,\tau,m}$ of fully discrete problem $(E_{h,\tau})$ is estimated as follows.*

(i) *For* $s \in [1/2, \infty)$,

$$\begin{aligned} &\| \varphi(m\tau) - \varphi_{h,\tau,m} \| \\ &\le \varepsilon(h)^s \cdot C_1 \cdot \Big\{ (m\tau + \underline{\alpha}^{-1/2}) \cdot E(m\tau\,;\, s) \\ &\qquad + \Big(1 + \frac{\gamma}{\sqrt{1-\gamma^2}}\Big) \cdot \| A^s \varphi^1 \| + \| A^{s-1} \psi(m\tau) \| \Big\} \\ &+ \tau^{2\min(1,s)} \cdot \frac{(m-1)\tau + \underline{\alpha}^{-1/2}}{\sqrt{1-\gamma^2}} \cdot C_2 \cdot E(m\tau\,;\, \min(1, s)) \\ &+ \beta\tau^2 \cdot \frac{\gamma}{\sqrt{1-\gamma^2}} \cdot \{ \| A\varphi^1 \| + \| \psi(0) \| \} \\ &+ \beta\tau^{2\min(1,s)} \cdot m\tau \cdot C_3 \cdot E(m\tau\,;\, \min(1, s)) \\ &+ \delta\tau \cdot \frac{(m-1)\tau}{\sqrt{1-\gamma^2}} \cdot C_4 \cdot D((m-1)\tau\,;\, 1/2) \end{aligned}$$

$$+ F(m\,;s), \qquad m = 2,3,\cdots.$$

(ii) *For* $s \in [0,1/2]$,

$$\|\varphi(m\tau) - \varphi_{h,\tau,m}\|$$

$$\leq \varepsilon(h)^s \cdot C_5 \cdot \Big\{(m\tau + \underline{\alpha}^{-1/2}) \cdot E(m\tau\,;s) + \Big(1 + \frac{\gamma}{\sqrt{1-\gamma^2}}\Big) \cdot \|A^s\varphi^1\| + \|A^{s-1}\psi(m\tau)\|\Big\}$$

$$+ \tau^{2s} \cdot \frac{(m-1)\tau}{\sqrt{1-\gamma^2}} \cdot \rho^{2(1-s)} \cdot C_6 \cdot E(m\tau\,;s)$$

$$+ \tau \cdot \frac{1}{\sqrt{1-\gamma^2}} \cdot C_7 \cdot \Big\{\|A^{1/2}\varphi^1\| + \|\varphi^0\| + (\tau + \underline{\alpha}^{-1/2}) \cdot \max_{0\leq r\leq\tau} \|\psi(r)\|\Big\}$$

$$+ \beta\tau^{2s} \cdot m\tau \cdot \rho^{2(1-s)} \cdot C_8 \cdot E(m\tau\,;s)$$

$$+ \delta\tau^{2s} \cdot \frac{(m-1)\tau}{\sqrt{1-\gamma^2}} \cdot \rho^{2(1/2-s)} \cdot C_9 \cdot D((m-1)\tau\,;s)$$

$$+ G(m\,;s), \qquad m = 2,3,\cdots.$$

Here $C_i (i = 1,2,\cdots,9)$ *are constants independent of parameters* τ, β, δ, γ, ρ *and* $\underline{\alpha}$, *and function* $\varepsilon(h)$, *and*

$$D(t\,;s) = \|A^{s+1/2}\varphi^1\| + \|A^s\varphi^0\| + \|A^{s-1/2}\psi(0)\| + t \cdot \max_{0\leq r\leq t} \|A^{s-1/2}\psi^{(1)}(r)\|,$$

$$E(t\,;s) = \|A^{s+1/2}\varphi^1\| + \|A^s\varphi^0\| + \|A^{s-1/2}\psi(0)\| + \|A^{s-1}\psi^{(1)}(0)\| + (t + \underline{\alpha}^{-1/2}) \cdot \max_{0\leq r\leq t} \|A^{s-1}\psi^{(2)}(r)\|,$$

$$F(m\,;s) = \Big(\frac{\underline{\alpha}^{-1} + \beta\tau^2}{1-\gamma^2}\Big)^{1/2} \Big\| P_h\Big\{\varphi^0 + \frac{\tau}{2}[\psi(0) - A\varphi^1]\Big\} - \varphi^0_{h,\tau}\Big\| + \Big(1 + \frac{\gamma}{\sqrt{1-\gamma^2}}\Big)\|P_h\varphi^1 - \varphi^1_{h,\tau}\| + \frac{(m-1)\tau}{\sqrt{\underline{\alpha}}\sqrt{1-\gamma^2}} \cdot \max_{1\leq j\leq m-1} \|P_h\psi(j\tau) - \psi_{h,\tau,j}\|,$$

$$G(m\,;s) = \frac{(m-1)\tau}{\sqrt{\underline{\alpha}}\sqrt{1-\gamma^2}} \cdot \max_{1\leq j\leq m-1} \|P_h\psi(j\tau) - \psi_{h,\tau,j}\| + \Big(1 + \frac{\gamma}{\sqrt{1-\gamma^2}}\Big)\|P_h\varphi^1 - \varphi^1_{h,\tau}\| + \Big(\frac{\underline{\alpha}^{-1} + \beta\tau^2}{1-\gamma^2}\Big)^{1/2} \|P_h\varphi^0 - \varphi^0_{h,\tau}\|.$$

Theorem 2 is an extension of Theorem 2 in [7], with respect to parameter s.

The proof is given by using Duhamel's representation for the solution of the second order linear evolution equation, with the aid of the following key lemmas which are proved in Matsuki and Ushijima [2].

For a real number s, we consider the condition:

$$(B_s) \qquad \| A_h^s P_h \varphi \| \leq C \| A^s \varphi \| \text{ for } \varphi \in D(A^s),$$

where the constant C may depend on s, but does not depend on $h \in (0, \bar{h}]$.

Lemma 1. *If $s \geq 0$, then the condition (ε_s) implies the condition (B_σ) for $\sigma \in [0,1+s]$ with the constant C independent of $\sigma \in [0,1+s]$.*

Lemma 2. *If $-1 \leq s \leq 0$ the condition $(A_{\varepsilon,s})$ implies the condition (B_σ) for $\sigma \in [s, 0]$ with the constant C independent of $\sigma \in [s, 0]$.*

4. Convergence of the solution of the discrete problem. The following theorem is obtained by applying the approximation theory for semigroups of linear operators in Ushijima [6] to the present case.

Theorem 3. *Fix $\gamma \in (0,1)$ independently of $h \in (0, \bar{h}]$. For $h \in (0, \bar{h}]$, set the value of τ so as to satisfy condition $(T_{2\gamma}^{\rho})$. Let T be positive number. Assume that, for any $\varphi \in X$, it holds*

$$\lim_{h \to 0} \| (A^{-1} - A_h^{-1} P_h) \varphi \| = 0.$$

Suppose that $\varphi^1 \in D(A^{1/2})$, $\varphi^0 \in X$, $\phi \in C([0, \infty) : X)$. Let $\varphi(t)$, and $\{\varphi_{h,\tau,m} : m = 0,1,2,\cdots\}$, be the generalized solution of problem (E), and the solution of the problem $(E_{h,\tau})$, respectively. Then it holds

$$\lim_{h \to 0} \max_{0 \leq m\tau \leq T} \| \varphi_{h,\tau,m} - \varphi(m\tau) \| = 0.$$

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