49. *Singular Variation of Non-linear Eigenvalues. II*

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Let M be a bounded domain in R^3 with smooth boundary ∂M. Let w be a fixed point in M. Removing an open ball $B(\varepsilon\,;w)$ of radius ε with the center w from M, we get $M_\varepsilon = M\backslash\overline{B(\varepsilon\,;w)}$. For $p>1$ and $\varepsilon>0$ let $\lambda(\varepsilon)$ denote the positive number defined by

$$\lambda(\varepsilon)=\inf_{X_\varepsilon}\int_{M_\varepsilon}|\nabla u|^2\,dx, \tag{1.1$_\varepsilon$}$$

where $X_\varepsilon=\{u\in H_0^1(M_\varepsilon):\|u\|_{L^{p+1}(M_\varepsilon)}=1,\ u\geq 0\}$.
We consider the asymptotic behaviour of $\lambda(\varepsilon)$ as ε tends to 0. It is well known that there exists at least one positive solution u_ε which attains $(1.1)_\varepsilon$ in case of $p\in(1,5)$. We note that the minimizer satisfies $-\Delta u_\varepsilon=\lambda(\varepsilon)u_\varepsilon^p$ in M_ε and $u_\varepsilon=0$ on ∂M_ε. We put

$$\lambda=\inf_X\int_M|\nabla u|^2\,dx,$$

where $X=\{u\in H_0^1(M):\|u\|_{L^{p+1}(M)}=1,\ u\geq 0\}$.

In this paper we show the following

Theorem 1. *Assume that the positive solution of $-\Delta u=\lambda u^p$ in M under the Dirichlet condition on ∂M is unique. Then, there exists a constant $p^*(M)>1$ such that for any $p\in(1,p^*(M))$ we have*

$$\lambda(\varepsilon)-\lambda=4\pi\varepsilon u(w)^2+o(\varepsilon) \tag{1.2}$$

as ε tends to zero.

Example. $M=B(r)$, the ball of radius r, satisfies the assumption of Theorem 1, as is seen in Gidas-Ni-Nirenberg [1, Theorem 1 and p. 224, 2.9]. See also Dancer [2, Theorem 5].

Theorem 1 follows from the following Theorems 2 and 3.

Theorem 2 (Ozawa [5]). *Fix $p\in(1,5)$. Assume that the positive solution of $-\Delta u=\lambda u^p$ in M under the Dirichlet condition on ∂M is unique. Moreover assume that* $\mathrm{Ker}(A+\lambda pu^{p-1})=\{0\}$, *where we denote A by the linear operator $H^2(M)\cap H_0^1(M)\ni u\to\Delta u\in L^2(M)$. Then,* (1.2) *holds.*

Theorem 3. *Assume that the positive solution of $-\Delta u=\lambda u^p$ in M under the Dirichlet condition on ∂M is unique. Then, there exists $p^*(M)>1$ such that* $\mathrm{Ker}(A+\lambda pu^{p-1})=\{0\}$ *holds for $p\in(1,p^*(M))$.*

We consider the eigenvalue problem (1.3).

$$\begin{aligned}-\Delta\varphi&=\mu u^{p-1}\varphi &&\text{in } M\\ \varphi&=0 &&\text{in } \partial M.\end{aligned}\tag{1.3}$$

Let $\mu_1^{(p)}$ ($\mu_2^{(p)}$, respectively) be the first (the second, respectively) eigenvalue of (1.3). Let $\varphi_1^{(p)}$ be the first eigenfunction of (1.3) which is normalized as

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$$\int_M u^{p-1}(\varphi_1^{(p)})^2\,dx = 1, \quad (\varphi_1^{(p)})(x) > 0, \quad x \in M.$$

We know that $0 < \mu_1^{(p)} < \mu_2^{(p)}$. Theorem 3 is a consequence of the following result.

Proposition 1. *If $p > 1$ is sufficiently close to one, then*

$$\mu_1^{(p)} = \lambda. \tag{1.4}$$

And

$$\lim_{p\to 1} \mu_2^{(p)} = \mu_2. \tag{1.5}$$

Here μ_2 is the second eigenvalue of the Laplacian $-\Delta$ in M under the Dirichlet condition on ∂M.

Theorem 3 follows from the inequality $\mu_1^{(p)} < p\lambda < \mu_2^{(p)}$. If p is sufficiently close to 1, the above inequality holds and $p\lambda$ is not an eigenvalue.

Proof of Proposition 1. We want to show $\mu_1^{(p)} = \lambda$. We know that

$$\begin{aligned}\mu_1^{(p)} &= \inf_{\varphi \neq 0} \left(\int_M |\nabla\varphi|^2 dx\right)\left(\int_M u^{p-1}\varphi^2 dx\right)^{-1} \\ &= \inf_{\|\varphi\|_{L^{p+1}(M)}=1} \text{(the same term as above)} \\ &\geq \inf_{\|\varphi\|_{L^{p+1}(M)}=1} \int_M |\nabla\varphi|^2\,dx = \lambda\end{aligned}$$

by using

$$\int_M u^{p-1}\varphi^2 dx \leq \|u\|_{p+1}^{(p-1)/(p+1)} \|\varphi\|_{p+1}^{2/(p+1)} = 1,$$

where $\|\ \|_q$ denotes the $L^q(M)$ norm. On the other hand,

$$\begin{aligned}\mu_1^{(p)} &\leq \left(\int_M |\nabla u|^2 dx\right)\left(\int_M u^{p+1}\,dx\right)^{-1} \\ &= \int_M |\nabla u|^2\,dx = \lambda.\end{aligned}$$

Therefore, we get (1.4).

We can prove (1.5) by using the standard perturbation theory of linear operators. See Kato [3]. Thus, we get Proposition 1.

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