45. On Integrated Semigroups which are not Exponentially Bounded

By Isao MIYADERA,*) Masashi OKUBO,**) and Naoki TANAKA***)

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1. Introduction. Recently, as a generalization of the notion of exponentially bounded n-times integrated semigroups, Hieber [4] introduced that of exponentially bounded α-times integrated semigroups for positive numbers α and obtained interesting results by using Laplace transform techniques. But there exist integrated semigroups which are not exponentially bounded (and do not have the Laplace transforms) (see [5]). It is interesting to study the theory of α-times integrated semigroups which are not necessarily exponentially bounded. In this direction, some results in the special case where α is a nonnegative integer are found in Tanaka and Okazawa [6] and Thieme [7].

In this paper we deal with α-times integrated semigroups which are not necessarily exponentially bounded on a Banach space X for $\alpha \geq 0$. It should be noted that Laplace transform techniques are not available in our case. In §2 we investigate the basic properties of an α-times integrated semigroup and its generator. In §3 we give a characterization of the generator of an α-times integrated semigroup in terms of the associated abstract Cauchy problem. Applying this characterization we prove in §4 the following: (I) (Perturbation Theorem) If A generates an n-times integrated semigroup and if $B \in B(X)$ and $R(B)$ (the range of B) $\subset D(A^n)$ then $A + B$ generates an n-times integrated semigroup. (II) (Adjoint Theorem) If A is the densely defined generator of an α-times integrated semigroup then the adjoint A^* of A generates a β-times integrated semigroup on the adjoint X^* of X for every $\beta > \alpha$. These extend [2, Corollary 3.5] and [4, Corollary 3.7]. The proofs of main results are sketched here, and the details will be published elsewhere.

2. α-times integrated semigroups. Let X be a Banach space with norm $\|\cdot\|$. We denote by $B(X)$ the set of all bounded linear operators from X into itself. Generalizing [1, Definition 3.2] we introduce

Definition 2.1. Let α be a positive number. A family $\{U(t) : t \geq 0\}$ in $B(X)$ is called an *α-times integrated semigroup* on X, if

(a_1) $U(\cdot)x : [0, \infty) \to X$ is continuous for every $x \in X$,

(a_2) $$U(t)U(s)x = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}U(r)xdr - \int_0^s (t+s-r)^{\alpha-1}U(r)xdr\right)$$

for $x \in X$ and $t, s \geq 0$, where $\Gamma(\cdot)$ denotes the gamma function,

*) Department of Mathematics, School of Education, Waseda University.
**) Department of Mathematics, Nirasaki High School.
***) Department of Mathematics, Faculty of Science, Okayama University.

(a_3) $U(t)x = 0$ for all $t > 0$ implies $x = 0$.

For convenience we call a semigroup of class (C_0) on X also 0-*times integrated semigroup* on X.

Definition 2.2. Let $\{U(t) : t \geq 0\}$ be an α-times integrated semigroup on X, where $\alpha \geq 0$. The *generator* A of $\{U(t) : t \geq 0\}$ is defined as follows: $x \in D(A)$ and $Ax = y$ if and only if $U(t)x = \int_0^t U(r)ydr + \frac{t^\alpha}{\Gamma(\alpha+1)}x$ for $t \geq 0$.

Remark 2.1. When $\alpha = 0$, our definition of the generator coincides with that of the infinitesimal generator of a semigroup of class (C_0).

Proposition 2.1. *Let A be the generator of an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, where $\alpha \geq 0$. Then A is a closed linear operator in X, and we have*:

(2.1) $U(t)x \in D(A)$ *and* $AU(t)x = U(t)Ax$ *for* $x \in D(A)$ *and* $t \geq 0$,

(2.2) $\int_0^t U(r)xdr \in D(A)$ *and* $A\int_0^t U(r)xdr = U(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x$ *for* $x \in X$ *and* $t \geq 0$.

Proposition 2.2. *Let $\{V(t) : t \geq 0\}$ be a family in $B(X)$ such that $V(\cdot)x : [0, \infty) \to X$ is continuous for every $x \in X$, and let B be a closed linear operator in X. Let $\alpha \geq 0$. If $\{V(t) : t \geq 0\}$ satisfies two conditions*

(i) $\int_0^t V(s)xds \in D(B)$ *and* $V(t)x = B\int_0^t V(s)xds + \frac{t^\alpha}{\Gamma(\alpha+1)}x$ *for* $x \in X$ *and* $t \geq 0$,

(ii) $V(t)x = \int_0^t V(s)Bxds + \frac{t^\alpha}{\Gamma(\alpha+1)}x$ *for* $x \in D(B)$ *and* $t \geq 0$,

then there exists an $\omega > 0$ such that $(\omega, \infty) \subset \rho(B)$ (the resolvent set of B).

Sketch of proof. Let $\tau > 0$ be fixed. For $\lambda > 0$ we define $R_\tau(\lambda) \in B(X)$ by $R_\tau(\lambda)x = \lambda^\alpha \int_0^\tau e^{-\lambda t}V(t)xdt$ for $x \in X$. Using the identity

$$R_\tau(\lambda)x = \lambda^\alpha e^{-\lambda\tau}\left(\int_0^\tau V(s)xds\right) + \lambda^{\alpha+1}\int_0^\tau e^{-\lambda t}\left(\int_0^t V(s)xds\right)dt$$

we deduce from the condition (i) that $R_\tau(\lambda)x \in D(B)$ and

$$BR_\tau(\lambda)x = \lambda^\alpha e^{-\lambda\tau}\left(V(\tau)x - \frac{\tau^\alpha}{\Gamma(\alpha+1)}x\right) + \lambda^{\alpha+1}\int_0^\tau e^{-\lambda t}\left(V(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\right)dt$$

for $x \in X$ and $\lambda > 0$. Hence

(2.3) $(\lambda I - B)R_\tau(\lambda)x = (I - Q_\tau(\lambda))x$ for $x \in X$ and $\lambda > 0$,

where $Q_\tau(\lambda)x = \lambda^\alpha e^{-\lambda\tau}\left(V(\tau)x - \frac{\tau^\alpha}{\Gamma(\alpha+1)}x\right) + \frac{1}{\Gamma(\alpha+1)}\int_{\lambda\tau}^\infty e^{-t}t^\alpha dt \cdot x$ for $x \in X$ and $\lambda > 0$. Combining (i) and (ii) yields $B\int_0^t V(s)xds = \int_0^t V(s)Bxds$ for $x \in D(B)$ and $t \geq 0$. Using this fact and the closedness of B we see that $R_\tau(\lambda)Bx = BR_\tau(\lambda)x$ for $x \in D(B)$, and then

(2.4) $R_\tau(\lambda)(\lambda I - B)x = (I - Q_\tau(\lambda))x$ for $x \in D(B)$ and $\lambda > 0$.

Since $\| Q_\tau(\lambda) \| \to 0$ as $\lambda \to \infty$, we can choose an $\omega > 0$ such that $\| Q_\tau(\lambda) \| < 1$ for $\lambda > \omega$. Hence $(I - Q_\tau(\lambda))^{-1} \in B(X)$ if $\lambda > \omega$. This fact together with the relations (2.3) and (2.4) shows $(\omega, \infty) \subset \rho(B)$.

As a direct consequence of Propositions 2.1 and 2.2 we have

Corollary 2.3. *Let $\alpha \geq 0$. If A is the generator of an α-times integrated semigroup on X, then there exists an $\omega > 0$ such that $(\omega, \infty) \subset \rho(A)$.*

Proposition 2.4. *Let $\alpha \geq 0$. (1) Let A be the generator of an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X. If $u \in C([0, T]; X)$ satisfies $u(t) = A\int_0^t u(s)ds$ for $0 \leq t \leq T$, then $u(t) = 0$ for $0 \leq t \leq T$. (2) Every α-times integrated semigroup is uniquely determined by its generator.*

Definition 2.3. If an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, where $\alpha \geq 0$, satisfies the exponential growth condition

(a_4) there exist positive numbers M and a such that $\| U(t) \| \leq Me^{at}$ for $t \geq 0$,

then it is called an *exponentially bounded α-times integrated semigroup* on X.

Proposition 2.5. *Let $\{U(t) : t \geq 0\}$ be an α-times integrated semigroup on X satisfying the condition (a_4), and let A be the generator of $\{U(t) : t \geq 0\}$. Then $(a, \infty) \subset \rho(A)$ and $R(\lambda ; A)x = \lambda^\alpha \int_0^\infty e^{-\lambda t} U(t)x\,dt$ for $x \in X$ and $\lambda > a$.*

Sketch of proof. Since A is closed we deduce from (2.2) that

$$A\left(\lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}\left(\int_0^t U(s)xds\right)dt\right) = \lambda^{\alpha+1}\int_0^\infty e^{-\lambda t}\left(U(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\right)dt\,;$$

hence $(\lambda I - A)\left(\lambda^\alpha \int_0^\infty e^{-\lambda t} U(t)xdt\right) = x$ for $x \in X$ and $\lambda > a$. Combining this and (2.1) we have $\lambda^\alpha \int_0^\infty e^{-\lambda t} U(t)(\lambda I - A)xdt = x$ for $x \in D(A)$ and $\lambda > a$.

Remark 2.2. Proposition 2.5 shows that if an α-times integrated semigroup is exponentially bounded then our definition of the generator coincides with that due to Hieber [4].

3. Abstract Cauchy problems. Let A be a closed linear operator in X and $x \in X$. We consider the following abstract Cauchy problem (for A):

(ACP ; x) $\quad u'(t) = Au(t)$ for $t \geq 0$, and $u(0) = x$.

By a *classical solution* u to (ACP ; x) we mean that $u \in C^1([0, \infty); X)$ and $u(t)$ satisfies the above equation (ACP ; x). Since A is closed, u is a classical solution to (ACP ; x) if and only if $u \in C^1([0, \infty); X)$ and $u(t)$ satisfies the following integral equation

(ACP ; x)$_0$ $\quad u(t) = A\int_0^t u(s)ds + x$ for $t \geq 0$.

Now, let $\alpha \geq 0$ and consider the following integral equation which is the α-times integral version of (ACP ; x)$_0$:

(ACP ; x)$_\alpha$ $\quad u(t) = A\int_0^t u(s)ds + \dfrac{t^\alpha}{\Gamma(\alpha+1)}x$ for $t \geq 0$.

Definition 3.1. If $u \in C([0, \infty) ; X)$ and $u(t)$ satisfies $(\mathrm{ACP} ; x)_\alpha$ then u is called a *solution* to $(\mathrm{ACP} ; x)_\alpha$.

The following theorem extends and improves [7, Theorem 6.2].

Theorem 3.1. *Let $\alpha \geq 0$. An operator A is the generator of an α-times integrated semigroup on X if and only if A is a closed linear operator in X and the* $(\mathrm{ACP} ; x)_\alpha$ *has a unique solution for every $x \in X$.*

Sketch of proof. If A is the generator of an α-times integrated semigroup $\{U(t) : t \geq 0\}$ on X, then it follows from Propositions 2.1 and 2.4 (1) that A is a closed linear operator in X and $U(\cdot)x$ is a unique solution to $(\mathrm{ACP} ; x)_\alpha$ for every $x \in X$. To prove the converse, let $u(\cdot ; x)$ be the unique solution to $(\mathrm{ACP} ; x)_\alpha$. For $t \geq 0$ we define $U(t) : X \to X$ by $U(t)x = u(t ; x)$ for $x \in X$. Clearly, each $U(t)$ is a linear operator in X. To show $U(t) \in B(X)$ we consider an F-space (in the sense of [3]) $C([0, \infty) ; X)$ with the quasi-norm $\sum_{k=1}^{\infty} \| v \|_k / 2^k (1 + \| v \|_k)$ for $v \in C([0, \infty) ; X)$, where $\| v \|_k = \max \{\| v(t) \| : 0 \leq t \leq k\}$ for $k = 1,2,3, \cdots$ and a linear operator $T : X \to C([0, \infty) ; X)$ defined by $Tx = U(\cdot)x$ for $x \in X$. We then see that T is closed. By the closed graph theorem (see [3, Theorem 2.2.4]), T is continuous from X into $C([0, \infty) ; X)$. This shows that each $U(t) : X \to X$ is continuous, that is, $U(t) \in B(X)$. If $\alpha = 0$ then we see that $\{U(t) : t \geq 0\}$ is a semigroup of class (C_0). Next, we consider the case where $\alpha > 0$. It is clear that $\{U(t) : t \geq 0\}$ satisfies (a_1) and (a_3) in Definition 2.1. To show (a_2), let $s \geq 0$ and $x \in X$ be arbitrarily fixed and set

$$v(t) = \frac{1}{\Gamma(\alpha)} \left(\int_s^{t+s} (t + s - r)^{\alpha - 1} U(r)x dr - \int_0^t (t + s - r)^{\alpha-1} U(r)x dr \right)$$

for $t \geq 0$. Then we see that $A \int_0^\tau v(t) dt = v(\tau) - \frac{\tau^\alpha}{\Gamma(\alpha + 1)} U(s)x$ for $\tau \geq 0$, which means that v is a solution to $(\mathrm{ACP} ; U(s)x)_\alpha$ and hence (by the uniqueness of solutions) $v(t) = u(t ; U(s)x) = U(t) U(s)x$ for $t \geq 0$. Thus (a_2) is satisfied. To show that A is the generator of the α-times integrated semigroup $\{U(t) : t \geq 0\}$, let B be the generator of $\{U(t) : t \geq 0\}$. Let $x \in D(A)$ and set $w(t) = \int_0^t u(s ; Ax) ds + \frac{t^\alpha}{\Gamma(\alpha + 1)} x$ for $t \geq 0$. Since $u(s ; Ax) = Aw(s)$ for $s \geq 0$, we see that w is a solution to $(\mathrm{ACP} ; x)_\alpha$. The uniqueness of solutions shows $w(t) = u(t ; x)$, namely

$$(3.1) \quad U(t)x = \int_0^t U(s)Ax ds + \frac{t^\alpha}{\Gamma(\alpha + 1)} x \quad \text{for } x \in D(A) \text{ and } t \geq 0.$$

By (3.1) and $U(t)z = A \int_0^t U(r) z dr + \frac{t^\alpha}{\Gamma(\alpha + 1)} z$ for $z \in X$ and $t \geq 0$, we deduce from Proposition 2.2 that $(\omega, \infty) \subset \rho(A)$ for some $\omega > 0$. Combining this fact and Corollary 2.3, we obtain $\rho(A) \cap \rho(B) \neq \emptyset$. From this fact and the relation that $A \subset B$ (by (3.1)) it follows that $A = B$.

In the special case where α is a nonnegative integer we may prove

Theorem 3.2. *Let α be a nonnegative integer. Then the equivalent conditions in Theorem* 3.1 *are equivalent to the statement that A is a closed linear operator in X with nonempty resolvent set and the* $(\mathrm{ACP} ; x)$ *has a unique clas-*

sical solution for every $x \in D(A^{\alpha+1})$.

4. Applications. This section is devoted to applications of Theorem 3.1. We start with the following perturbation theorem.

Theorem 4.1. *Let A be the generator of an n-times integrated semigroup $\{U(t): t \geq 0\}$ on X, where n is a nonnegative integer. If $B \in B(X)$ and $R(B) \subset D(A^n)$, then $A+B$ is the generator of an n-times integrated semigroup on X. In the special case where $\{U(t): t \geq 0\}$ satisfies the condition* (a_4), *the n-times integrated semigroup $\{V(t): t \geq 0\}$ generated by $A+B$ satisfies the estimate that $\| V(t) \| \leq Me^{(a+K)t}$ for $t \geq 0$, where $K = M \| A^n B \| + \max_{0 \leq k \leq n-1} (\| A^k B \| / a^k)$.*

Sketch of proof. We shall consider the case where $n \geq 1$. Let $x \in X$ and $T \in (0, \infty)$ be arbitrarily fixed. We consider the Banach space $C([0, T]; X)$ with supremum norm and define an operator $W: C([0, T]; X) \to C([0, T]; X)$ by

$$(Wf)(t) = U(t)x + (d^n/dt^n) \int_0^t U(t-s)Bf(s)\,ds$$

$$\left(= U(t)x + \int_0^t U(t-s)A^n Bf(s)\,ds + \int_0^t \sum_{k=0}^{n-1} \frac{(t-s)^k}{k!} A^k Bf(s)\,ds\right)$$

for $f \in C([0, T]; X)$. Here we note that W is well-defined since $A^k B \in B(X)$ for every k with $0 \leq k \leq n$, by the closed graph theorem. Then, the fixed point theorem asserts that W has a unique fixed point. Therefore it is seen that for every $x \in X$ and $T > 0$ there exists a unique element $v_{x,T} \in C([0, T]; X)$ such that $v_{x,T}(t) = U(t)x + (d^n/dt^n) \int_0^t U(t-s)Bv_{x,T}(s)\,ds$ for $0 \leq t \leq T$. Now, for each $x \in X$ we define $v_x: [0, \infty) \to X$ by $v_x(t) = v_{x,T}(t)$ if $0 \leq t \leq T$. Then v_x is a unique element in $C([0, \infty); X)$ satisfying

$$v_x(t) = U(t)x + (d^n/dt^n) \int_0^t U(t-s)Bv_x(s)\,ds$$

for $t \geq 0$. We see that for every $x \in X$, v_x is the unique solution to $(\mathrm{ACP}; x)_n$ for the operator $A+B$. Clearly, $A+B$ is a closed linear operator in X. By virtue of Theorem 3.1, $A+B$ is the generator of an n-times integrated semigroup $\{V(t): t \geq 0\}$ defined by $V(t)x = v_x(t)$ for $x \in X$ and $t \geq 0$.

Finally, suppose that $\{U(t): t \geq 0\}$ satisfies the condition (a_4). We find the estimate that $\| V(t)x \| \leq Me^{at} \| x \| + K \int_0^t e^{a(t-s)} \| V(s)x \|\,ds$, namely

$$e^{-at} \| V(t)x \| \leq M \| x \| + K \int_0^t e^{-as} \| V(s)x \|\,ds$$

for $x \in X$ and $t \geq 0$, where $K = M \| A^n B \| + \max_{0 \leq k \leq n-1} (\| A^k B \| / a^k)$. The Gronwall inequality shows $e^{-at} \| V(t)x \| \leq Me^{Kt} \| x \|$ for $x \in X$ and $t \geq 0$.

As an another application we give the following adjoint theorem.

Theorem 4.2. *Let A be the densely defined generator of an α-times integrated semigroup $\{U(t): t \geq 0\}$ on X, where $\alpha \geq 0$. Then we have:*
(i) *The adjoint A^* of A is the generator of an $(\alpha+\gamma)$-times integrated semigroup on the adjoint X^* of X for every $\gamma > 0$.* (ii) $\{U(t)^*|_{\overline{D(A^*)}}: t \geq 0\}$ *is an*

α-times integrated semigroup on $\overline{D(A^*)}$ whose generator is the part of A^* in $\overline{D(A^*)}$, where $U(t)^*|_{\overline{D(A^*)}}$ denotes the restriction of $U(t)^*$ to $\overline{D(A^*)}$.

Sketch of proof for (i). Let $\gamma > 0$ and define $V(t) \in B(X)$ for $t \geq 0$ by $V(t)x = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} U(s)xds$ for $x \in X$. It is seen that $V(\cdot)^*x^* \in C([0, \infty); X^*)$ for every $x^* \in X^*$. By Theorem 3.1 we see that A is the generator of the $(\alpha+\gamma)$-times integrated semigroup $\{V(t) : t \geq 0\}$ on X; hence $\int_0^t V(r)Axdr = V(t)x - \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}x$ for $x \in D(A)$ and $t \geq 0$. Using this we have

$$\left(Ax, \int_0^t V(r)^*x^*dr\right) = \left(\int_0^t V(r)Axdr, x^*\right) = \left(x, V(t)^*x^* - \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}x^*\right)$$

for $x \in D(A)$, $x^* \in X^*$ and $t \geq 0$. From the definition of the adjoint we deduce that $\int_0^t V(r)^*x^*\,dr \in D(A^*)$ and $A^*\int_0^t V(r)^*x^*dr = V(t)^*x^* - \frac{t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}x^*$ for $x^* \in X^*$ and $t \geq 0$. This means that for every $x^* \in X^*$, $V(\cdot)^*x^*$ is a solution to $(\mathrm{ACP}; x^*)_{\alpha+\gamma}$ for the operator A^*. To show the uniqueness of solutions, let $u^*(\cdot)$ be a solution to $(\mathrm{ACP}; x^*)_{\alpha+\gamma}$ and set $w^*(t) = V(t)^*x^* - u^*(t)$ for $t \geq 0$. Then we see that $w^*(\cdot) \in C([0, \infty); X^*)$ and $w^*(t) = A^*\int_0^t w^*(r)dr$ for $t \geq 0$. Combining this with $(d/ds)V(t-s)x = -AV(t-s)x - \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}x$ for $x \in D(A)$ and $0 \leq s \leq t$, we find

$$(d/ds)\left(V(t-s)x, \int_0^s w^*(r)dr\right) = -\left(x, \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\int_0^s w^*(r)dr\right)$$

for $x \in D(A)$ and $0 \leq s \leq t$. Integrating this over $[0, t]$ and noting $V(0) = 0$, we see that $\left(x, \int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\left(\int_0^s w^*(r)dr\right)ds\right) = 0$ for $x \in D(A)$ and $t \geq 0$. Since $D(A)$ is dense in X we have $\int_0^t \frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\left(\int_0^s w^*(r)dr\right) ds = 0$; hence $w^*(t) = 0$ for $t \geq 0$. By Theorem 3.1, $\{V(t)^* : t \geq 0\}$ is an $(\alpha+\gamma)$-times integrated semigroup on X^* whose generator is A^*.

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