42. *Fully Idempotent Semirings*

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In this paper A will denote a semiring $(A, +, \cdot)$ as defined, for example, in [2], that is, two semigroups $(A, +)$ and $(A, \cdot)$ such that addition distributes over multiplication. Moreover, we shall always assume that $(A, +)$ is commutative and $(A, +, \cdot)$ has an absorbing zero 0, that is, $a + 0 = 0 + a = a$ and $0 \cdot a = a \cdot 0 = 0$ hold for all $a \in A$. The notions of left, right, and two-sided ideals, as well as sums and products of such ideals are defined as usual. The word ideal will always mean a two-sided ideal. An ideal P of A is called *prime* (*irreducible*; *strongly irreducible*) if $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P (I \cap J = P \Rightarrow I = P$ or $J = P$; $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P)$ holds for all ideals I, J of A. Thus any prime ideal is strongly irreducible and any strongly irreducible ideal is irreducible. A semiring A is called *fully idempotent* if each (two-sided) ideal of A is idempotent (an ideal I is idempotent if $I^2 = I$). A semiring A is called (*von Neumann*) *regular* if $x \in xAx$, for all $x \in A$ (cf. [8,10]). Regular semirings and simple semirings (i. e. having no non-zero proper ideals) form proper subclasses of fully idempotent semirings. Below we characterize fully idempotent semirings by the property that each ideal is the intersection of those prime ideals which contain it. We also obtain a similar characterization of semisimple semigroups, that is, semigroups all of whose ideals are idempotent.

We begin with the following result which is due to Courter [4]. Courter, in fact, proved this result for rings instead of semirings. However, an examination of his proof shows that it works in the case of semirings.

Proposition 1. *The following assertions for a semiring A are equivalent*:

1. *A is fully idempotent*;
2. *for each pair of ideals I, J of A, $I \cap J = IJ$*;
3. *for each right ideal R and two-sided ideal I, $R \cap I \subseteq IR$*;
4. *for each left ideal L and two-sided ideal I, $L \cap I \subseteq LI$.*

Recall that the lattice of ideals of a semiring is not, in general, distributive or even modular (cf. [7]). Below, we show that the ideal lattice of a fully idempotent semiring is a complete Brouwerian and hence distributive lattice. A lattice $\mathscr{L}$ is called *Brouwerian* if, for any $a, b \in \mathscr{L}$, the set of all $x \in \mathscr{L}$ satisfying $a \wedge x \leq b$ contains a greatest element c, the *pseudo-complement* of a relative to b.

Proposition 2. *If A is a fully idempotent semiring, then the ideal lattice $\mathscr{L}_A$ of A is a complete Brouwerian lattice.*

Proof. Clearly, $\mathscr{L}_A$ is a complete lattice under the sum and intersection of ideals. Let B and C be ideals of A. By Zorn's lemma, there is an ideal M of A which is maximal in the family of ideals I satisfying $B \cap I \subseteq C$. Thus if I is any such ideal then $BI \subseteq C$, and moreover, an easy calculation shows $B(I + M) \subseteq C$. Hence $B \cap (I + M) \subseteq C$, by the above proposition. By the maximality of M we get $I + M = M$, and therefore $I \subseteq M$, as we were to show.

Corollary 1. *$\mathcal{L}_A$ satisfies the infinite meet distributive law:*

$$I \cap (\sum_\alpha I_\alpha) = \sum_\alpha (I \cap I_a).$$

Proof. Follows from ([1], V, Thm 24).

Corollary 2. *$\mathcal{L}_A$ is distributive.*

Proof. Follows from ([1], II. 11).

The following proposition shows that the concepts of prime ideals, irreducible ideals and strongly irreducible ideals coincide for fully idempotent semirings.

Proposition 3. *Let A be a fully idempotent semiring. Then the following assertions for an ideal P of A are equivalent:*

1. *P is irreducible;*
2. *P is prime.*

Proof. As (2) $\Rightarrow$ (1) is clear, it suffices to show that (1) $\Rightarrow$ (2). Suppose $IJ \subseteq P$ for ideals I, J of A. Hence $I \cap J \subseteq P$, by Prop. 1. Thus it follows that $(I \cap J) + P = P$. Since the ideal lattice of A is distributive by Corollary 2 of the above proposition, we have $P = (I \cap J) + P = (I + P) \cap (J + P)$. Since P is irreducible, so $I + P = P$ or $J + P = P$. This implies that $I \subseteq P$ or $J \subseteq P$. Hence P is a prime ideal.

Theorem 1. *Let A be a semiring. Then the following assertions are equivalent:*

1. *A is fully idempotent;*
2. *each proper ideal of A is the intersection of prime ideals which contain it.*

Proof (1) $\Rightarrow$ (2). Let I be a proper ideal of A and let $\{P_\alpha : \alpha, \in \Lambda\}$ be a family of prime ideals of A which contain I. Clearly, $I \subseteq \cap P_\alpha$. To prove the converse, suppose $a \notin I$. By Zorn's lemma, there exists an ideal P_α such that P_α is proper, $I \subseteq P_\alpha$, $a \notin P_\alpha$, and P_α is maximal with these properties. Then P_α is irreducible. For, suppose on the contrary, $P_\alpha = K \cap L$, and both K and L properly contain P_α. Then K and L both contain a. Hence $a \in K \cap L$. This contradicts the assumption that $P_\alpha = K \cap L$. Hence P_α is irreducible, and so it is prime by Prop. 3. This establishes the existence of a prime ideal P_α such that $a \notin P_\alpha$ and $I \subseteq P_\alpha$. Hence $a \notin \cap P_\alpha$. As this is true for every $a \notin I$, the desired result follows. We now prove that (2) $\Rightarrow$ (1). Let I be any ideal of A. If $I^2 = A$ then I is certainly idempotent. If $I^2 \neq A$ then I^2 is a proper ideal of A and so it is the intersection of prime ideals P_α of A, by our assumption. Hence $I^2 = \cap P_\alpha \subseteq P_\alpha$, for each α. This implies that $I \subseteq P_\alpha$ for every α, since P_α is a prime ideal. Thus we have $I \subseteq \cap_\alpha P_\alpha = I^2$. Hence $I = I^2$, and so A is fully idempotent.

Let $\mathcal{P}_A$ denote the set of proper prime ideals of A. For any ideal I of A, define $\Theta_I = \{J \in \mathscr{P}_A : I \not\subseteq J\}$ and $\tau(\mathscr{P}_A) = \{\Theta_I : I$ is an ideal of $A\}$. The set $\mathscr{P}_A$ may be topologized in a manner analogous to the construction of prime spectrum of rings. More precisely, we may state the following theorem.

Theorem 2. *Let A be a fully idempotent semiring. The set $\tau(\mathscr{P}_A)$ forms a topology on the set $\mathscr{P}_A$ and the assignment $I \mapsto \Theta_I$ is an isomorphism between the lattice $\mathscr{L}_A$ of ideals of A and the lattice of open subsets of $\tau(\mathscr{P}_A)$.*

We now construct a class of examples of fully idempotent semirings which are neither regular nor simple.

Example 1. Let S be a semigroup with identity e and let C denote the bicyclic semigroup, that is, $C = N_0 \times N_0$; where N_0 is the set of non-negative integers and the multiplication in C is defined by

$$(m, n)(p, q) = (m + p - \min(n, p), n + q - \min(n, p)).$$

It is well known (see, e. g. [3], 1.12) that C is a bisimple inverse monoid with $(m, n)^{-1} = (n, m)$ and identity $(0, 0)$. Let $W = C \times S$ with the following multiplication:

$$((m, n), s)(p, q), t) = ((m, n)(p, q), f(n, p))$$

where $f(n, p) = s$, t or st according to whether $n > p$, $n < p$ or $n = p$. It can be verified that W is a simple semigroup with $((0, 0), e)$ as its identity element. Furthermore, W is regular if and only if S is regular. Let $W = C \times S$, where S is any non-regular monoid (e. g. $S = (N, \cdot)$). Define $A = W \cup \{\infty\}$, where $\{\infty\}$ is the ring with a single element. Define $w_1 + w_2 = \infty = w_1 + \infty = \infty + w_2$; $w_1 \cdot w_2 =$ product in W and $w_1 \cdot \infty = \infty \cdot w_2 = \infty$ for $w_1, w_2 \in W$. Now adjoin an absorbing zero $0 \notin A$ to A. Then $(A \cup \{0\}, +, \cdot)$ is a semiring with 0 as the additive identity and multiplicative zero. This semiring is not regular, and its only non-zero proper ideal is $I = \{0\} \cup \{\infty\}$, which is idempotent. Hence $(A \cup \{0\}, +, \cdot)$ has the required properties.

In the rest of this paper we consider semigroups all of whose ideals are idempotent. Such semigroups, called *semisimple semigroups,* admit many characterizations (see, e. g. [3] vol. I, p. 76). Below we characterize this class of semigroups by the property that each proper ideal is the intersection of prime ideals which contain it. The notions of idempotent, prime, irreducible and strongly irreducible ideals are extended to semigroups in a natural way and so are not defined explicitly. For the usual terminologies in semigroups, we refer to [3].

Theorem 3. *Let S be a semigroup. Then the following assertions are equivalent:*

1. *S is semisimple;*
2. *each proper ideal of S is the intersection of prime ideals which contain it;*
3. *no Rees factor semigroup of S contains a non-zero nilpotent ideal;*
4. *the ideal lattice $\mathscr{L}_S$ of S is a distributive lattice with $I \cap J = IJ$, for all I, $J \in \mathscr{L}_S$.*

If S is assumed commutative, then the above assertions are equivalent to:

5. *S is (von Neumann) regular.*

Proof (1)$\Rightarrow$(2). First we show that each proper ideal of S is contained in a proper irreducible ideal. Let K be a proper ideal of S and let $s \in S \setminus K$. Let P_s be any ideal maximal with respect to $K \subseteq P_s$ but $s \notin P_s$. Suppose that $P_s = A \cap B$, where A and B are ideals of S with $A \neq P_s$ and $B \neq P_s$. The maximality of P_s requires that $s \in A$ and $s \in B$. But then $s \in A \cap B = P_s$, which is a contradiction. Hence P_s is irreducible. Let $\{P_s : s \in S \setminus K\}$ be the family of proper irreducible ideals containing K. Then $K \subseteq \cap P_s$. For the reverse inclusion, let $t \in S$ such that $t \notin K$. Then as argued above, there exists an irreducible ideal P_t containing K such that $t \in S \setminus K$ and $t \notin P_t$. This implies that $t \notin \cap_{s \in S \setminus K} P_s$. Hence, by contrapositivity, $\cap_{s \in S \setminus K} P_s \subseteq K$. Thus $K = \cap_{s \in S \setminus K} P_s$. We now show that each P_s is prime. If I and J are ideals of S satisfying $IJ \subseteq P_s$, then $(I \cap J)^2 \subseteq IJ \subseteq P_s$. Since S is semisimple, so $(I \cap J)^2 = I \cap J$. Thus $I \cap J \subseteq P_s$. This implies that $(I \cap J) \cup P_s = P_s$. Since $(I \cap J) \cup P_s = (I \cup P_s) \cap (J \cup P_s)$, so $(I \cup P_s) \cap (J \cup P_s) = P_s$. As P_s is irreducible, it follows that $I \cup P_s = P_s$ or $J \cup P_s = P_s$. Thus $I \subseteq P_s$ or $J \subseteq P_s$, showing that P_s is prime.

(2)$\Rightarrow$(1). Let I be any ideal of S. If $I^2 = S$ then I is clearly an idempotent ideal. If $I^2 \neq S$ then I^2 is a proper ideal, and so by the hypothesis, $I^2 = \cap_\alpha \{P_\alpha : P_\alpha$ is a prime ideal$\}$. Since each P_α is a prime ideal and $I^2 = \cap_\alpha P_\alpha \subseteq P_\alpha$, it follows that $I \subseteq P_\alpha$, for each α. Hence $I \subseteq \cap_\alpha P_\alpha = I^2$. This implies that $I = I^2$, and hence S is semisimple.

The equivalence of (1), (3), and (4) is easily deduced from Courter's main theorem ([4]) by making necessary modifications. Moreover, if S is commutative then it is easy to verify that (5)$\Rightarrow$(1) and (4)$\Rightarrow$(5) (cf.[3] vol. I p. 34, [5]).

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