41. *Center Curves in the Moduli Space of the Real Cubic Maps*

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1. Center curves. We consider the family of real cubic maps $x \mapsto g(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ ($c_3 \neq 0$, $c_i \in \boldsymbol{R}$). By a suitable real affine transformation, any map $g(x)$ is transformed to a unique map $f(x) = \sigma x^3 - 3Ax + \sqrt{|B|}$, where $\sigma := sgn(g''')$. The real affine conjugacy class of g or f can be represented by (A, B) if $B \neq 0$. But if $B = 0$, σ should be added as an essential class invariant, as $x \mapsto x^3 - 3Ax$ and $x \mapsto -x^3 - 3Ax$ belong to different classes. Milnor ([1]) defined thus the disjoint union of the upper half-plane $\boldsymbol{H}^+ = \{(A, B) \mid B \geq 0\}$ and the lower half-plane $\boldsymbol{H}^- = \{(A, B) \mid B \leq 0\}$ to be the **moduli space** of the conjugacy classes of our maps.

The map $x \mapsto f(x)$ has two critical points $\pm\sqrt{\sigma A}$ (which may coincide or be purely imaginary) which will be denoted with p_1, p_2. When the orbit $\{f^n(p_1), f^n(p_2); n = 1,2,\ldots\}$ is a finite set, f is called a **center map** and the coordinates (A, B) of f will be called a **center** in the moduli space.

Following Milnor ([1]), the centers are classified as follows. (In the following t, p, q denote integers.)

A center is of the type $\mathcal{A}_p$ if two critical points of the center map coincide $p_1 = p_2$ and has the period $p : f^p(p_1) = p_1$. (In fact, only possible values for p in this case are 1, 2.) A center is of the type $\mathcal{B}_{p+q}$ if $f^p(p_1) = p_2$ and $f^q(p_2) = p_1$; of the type $\mathcal{C}_{(t)q}$ if $f^t(p_1) = p_2$ and $f^q(p_2) = p_2$; of the type $\mathcal{D}_{p,q}$ if $f^p(p_1) = p_1$ and $f^q(p_2) = p_2$.

These exhaust all types of centers. It is clear that there are only a finite number of centers of a given type.

Example. There exist three centers of type $\mathscr{C}_{(3)1}$. The corresponding parameters are $(A, B) = (-.75040, -.18820)$, $(-.74949, -.18679)$, $(-.0924912, -.0614376)$.

From now on, we shall limit our consideration to the case $\sigma A > 0$. Then we observe that the following theorem holds.

Theorem. *For a given p, there exist an algebraic curve CDp containing all centers of the type $\mathcal{C}_{(k)p}$ and $\mathcal{D}_{k,p}$, and another algebraic curve BCp containing all centers of the type $\mathcal{B}_{p+k}$ and $\mathcal{C}_{(p)k}$. Precisely we obtain the following curves;*

$$\mathrm{CD1} : B = 4A\left(A + \frac{1}{2}\right)^2,$$

$$\mathrm{BC1} : B = 4A\left(A - \frac{1}{2}\right)^2,$$

$$\mathrm{CD2} : B^2 - 8A^3B + 4A^2B - 5AB + 2B + 16A^6 - 16A^5 - 12A^4 + 16A^3 - 4A + 1 = 0,$$

$$\text{BC2}: B^3 - 12A^3B^2 - 6AB^2 + 2B^2 + 48A^6B + 24A^3B + 21A^2B - 6AB + B - 64A^9 + 96A^7 - 20A^5 - 12A^3 - A = 0,$$

$\cdots : \cdots .$

Proof. We shall give a proof for CD1. For a center map $f(x) = \sigma x^3 - 3Ax + \sqrt{|B|}$ to be of the type $\mathcal{C}_{(k)1}$ or $\mathcal{D}_{1,k}$, we should have $f(\sqrt{\sigma A}) = \sqrt{\sigma A}$ or $f(-\sqrt{\sigma A}) = \sqrt{\sigma A}$, whence follows $B = 4A\left(A + \frac{1}{2}\right)^2$. In the same way, we obtain the curves CD_p and BC_p.

Remark 1. The centers of type $\mathcal{C}_{(k)1}$ and type $\mathcal{D}_{1,k}$ exist only in the third quadrant.

Remark 2. We can factorize the curve CD2 as follows;

$$\text{CD2-1}: 2B - (4A - 1)\sqrt{9A^2 - 4A} - 8A^3 + 4A^2 - 5A + 2 = 0,$$
$$\text{CD2-2}: 2B + (4A - 1)\sqrt{9A^2 - 4A} - 8A^3 + 4A^2 - 5A + 2 = 0.$$

The curves CD_p and BC_p are called **center curves.**

2. Monotonicity of topological entropy along center curves. In [2], Milnor and Thurston considered the growth number s and topological entropy $\log s$ of continuous maps f, and conjectures concerning them in case of cubic maps were enunciated by Milnor in [1]. Block and Keesling ([3]) gave then an algorithm to calculate them and Prof. Milnor kindly sent us their papers showing the result of calculation. (In these papers, a different representation is used for the moduli space. For $\boldsymbol{H}^+$ the coordinates (A, b), $b = \sqrt{B}$, instead of (A, B) and for $\boldsymbol{H}^-$ the coordinates (A, b'), $b' = -\sqrt{|B|}$, instead of (A, B) are used.)

Using the method of [2], we calculate growth numbers of cubic maps along center curves CD1, BC1, CD2-1, and CD2-2. The growth number is identically 1 on CD1 in the upper half (A, B)-plane and on CD2-1.

Center curves BC1 and CD2-2 are shown in Fig. 1 and CD1, BC1, and CD2-2 in Fig. 2 together with the equi-growth number lines in the figures due to Block and Keesling. The region of Fig. 1 (resp. 2) is $[.57, 1.03] \times [0, .43]$ (resp. $[-1.05, -.09] \times [0, -1.35]$) in $(A, b) -$ (resp. $(A, b') -$) plane.

A glance at Figs. 1 and 2 suggests that the growth number and the topological entropy vary monotonously along an center curve. We should like to propose this conjecture. Tables 1-5 which we have calculated support also this conjecture.

Bifurcation diagrams for the cubic maps along center curves are shown in Figs. 3 and 4. Fig. 3 corresponds to Table 2 and Fig. 4 to Table 5. That we see here flip bifurcations as in unimodal case seems to lend strong support to our conjecture.

We would like to express our gratitude to Dr. M. Shishikura for useful comments and enlightening discussions.

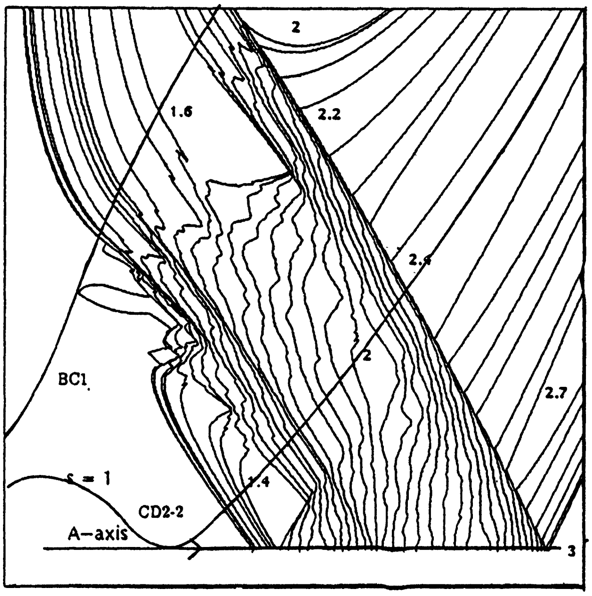
2
1.6
2.2
2.4
2
BC1
2.7
s = 1
1.4
CD2-2
A-axis
3

Fig. 1

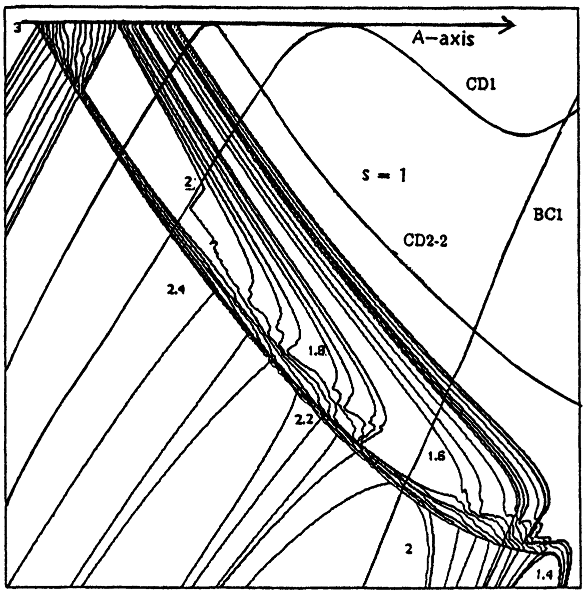
3
A-axis
CD1
s = 1
2
BC1
CD2-2
2.4
1.8
2.2
1.6
2
1.4

Fig. 2

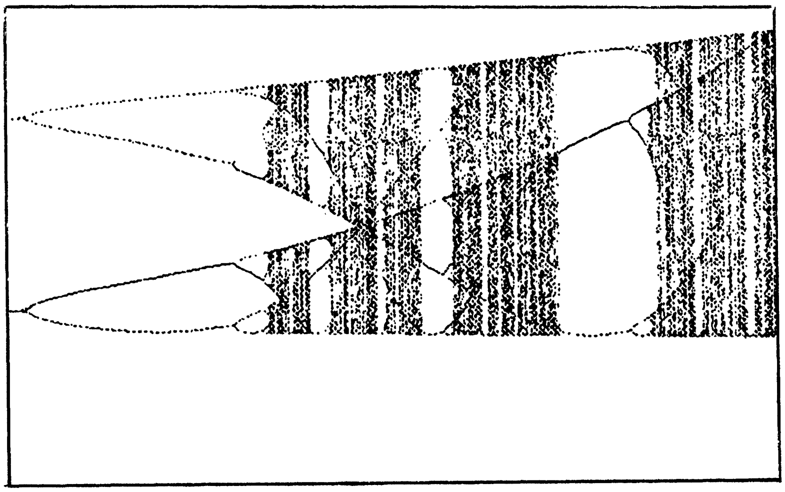

Fig. 3. Bifurcation diagrams for the cubic maps with a parameter A along center curve BC1 ($.6 < A < .74999$).

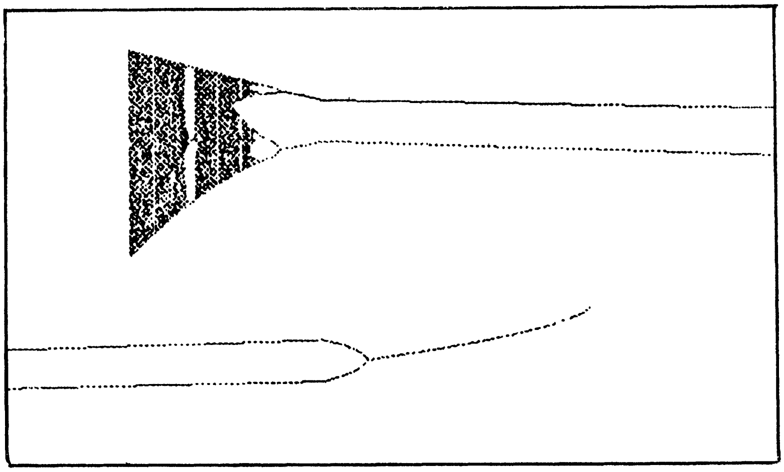

Fig. 4. Bifurcation diagrams for the cubic maps with a parameter A along center curve CD2 ($-.85 < A < -.5$).

Table

(A,B)	type	s	(A,B)	type	s
$(\star)$ indicates that parameter (A,B) is not a center.					
Table 1: CD1					
$\sim(-.7825,-.25)$	$(\star)$	$1+\sqrt{2}$	$(-.7,-.112)$	$(\star)$	1.7291
$(-.7820,-.2488)$	$\mathcal{C}_{(4)1}$	2.3593	$(-.6987,-.1104)$	$\mathcal{D}_{1,5}$	1.7156
$(-.7787,-.2420)$	$(\star)$	2.2226	$\sim(-.6974,-.1088)$	$(\star)$	
$(-.7773,-.2390)$	$\mathcal{D}_{1,3}$	2.2055	$(-.6887,-.0981)$	$\mathcal{D}_{1,6}$	$\frac{1+\sqrt{5}}{2}$
$(-.7762,-.2369)$	$\mathcal{C}_{(9)1}$	2.1903	$\sim(-.6861,-.0950)$	$\mathcal{D}_{1,3}$	
$(-.7749,-.2344)$	$(\star)$	2.1727	$(-.6737,-.0813)$	$\mathcal{D}_{1,5}$	1.5128
$(-.7637,-.2125)$	$\mathcal{C}_{(9)1}$	2	$(-.6524,-.0606)\sim$	$\mathcal{D}_{1,8}$	1
$\sim(-.727..,-.149..)$	$(\star)$				
Table 2: BC1 (in the upper half-plane)					
$\sim(.6458,.0549)$	$\mathcal{C}_{(1)8}$	1	$(.7083,.1229)$	$\mathcal{C}_{(1)3}$	$\frac{1+\sqrt{5}}{2}$
$(.6597,.0673)$	$(\star)$	1.2720	$(.7132,.1297)$	$\mathcal{B}_{1+2}$	
$(.6722,.0797)$	$\mathcal{C}_{(1)7}$	1.4655	$(.7375,.1664)$	$\mathcal{C}_{(1)7}$	1.7548
$(.6847,.0934)$	$(\star)$	1.5128	$(.7444,.1779)$	$\mathcal{C}_{(1)4}$	1.8393
$(.6986,.1102)$	$(\star)$	1.5972	$(.7446,.1782)$	$\mathcal{B}_{1\mp 3}$	
			$(\frac{3}{4},\frac{3}{16})\sim$	$(\star)$	2
Table 3: BC1 (in the lower half-plane)					
$\sim(-.2950,-.7457)$	$(\star)$	1	$(-.3875,-1.2208)$	$\mathcal{B}_{1+7}$	1.7291
$(-.3285,-.9019)$	$(\star)$	1.5302	$(-.3915,-1.2446)$	$\mathcal{C}_{(1)4}$	1.8392
$(-.3291,-.9052)$	$(\star)$		$(-.3925,-1.2506)$	$\mathcal{B}_{1+6}$	1.93823
$(-.3325,-.9217)$	$(\star)$	1.5302	$(-.3968,-1.2768)\sim$	$(\star)$	2
$(-.3533,-1.0291)$	$(\star)$	$\frac{1+\sqrt{5}}{2}$			
$(-.3646,-1.0904)$	$\mathcal{B}_{1+2}$				
$(-.3808,-1.1819)$	$(\star)$				
Table 4: CD2-2 (in the upper half-plane)					
$(.4444,.6708)$	$\mathcal{C}_{(9)2}$	1	$(.8152,.0115)$	$\mathcal{C}_{(6)2}$	1.8246
$\sim(.7443,.0009)$	$\mathcal{D}_{2,6}$		$(.8507,.0231)$	$(\star)$	2
$(.7528,.0015)$	$\mathcal{D}_{2,6}$	1.1884	$(.8536,.0243)$	$\mathcal{C}_{(2)2}$	
$(.7693,.0031)$	$\mathcal{C}_{(9)2}$	$\sqrt{2}$	$(.8861,.0402)$	$\mathcal{C}_{(9)2}$	
$(.7743,.0037)$	$\mathcal{C}_{(9)2}$		$(.8903,.0427)\sim$	$(\star)$	$1+\sqrt{2}$
$(.8069,.0095)$	$(\star)$	1.7653			
Table 5: CD2-2 (in the lower half-plane)					
$\sim(-.8571,-.1147)$	$(\star)$	$1+\sqrt{2}$	$(-.7733,-.0209)$	$\mathcal{D}_{2,9}$	1.6988
$(-.845,-.0959)$	$\mathcal{C}_{(9)2}$	2	$(-.7706,-.0192)$	$\mathcal{D}_{2,8}$	1.6483
$\sim(-.7966,-.0389)$	$(\star)$		$(-.7685,-.0179)$	$\mathcal{D}_{2,9}$	$\frac{1+\sqrt{5}}{2}$
$(-.7882,-.0317)$	$\mathcal{D}_{2,9}$	1.9015	$(-.7672,-.0171)$	$\mathcal{D}_{2,6}$	
$(-.7879,-.0315)$	$\mathcal{D}_{2,8}$	1.8949	$(-.7653,-.0160)$	$\mathcal{D}_{2,3}$	
$(-.7859,-.0299)$	$\mathcal{D}_{2,9}$	1.8784	$(-.7544,-.0105)$	$\mathcal{D}_{2,5}$	1.5128
$(-.7836,-.0281)$	$\mathcal{D}_{2,8}$	1.8392	$(-.7506,-.0088)$	$\mathcal{D}_{2,7}$	1.4655
$(-.7834,-.0280)$	$\mathcal{D}_{2,4}$		$(-.7437,-.0062)$	$\mathcal{D}_{2,6}$	1.2720
$(-.7752,-.0221)$	$\mathcal{D}_{2,5}$	1.7220	$(-.7367,-.0040)\sim$	$\mathcal{D}_{2,8}$	1

References

[1] J. Milnor: Remarks on iterated cubic maps. Preprint # 1990/6, SUNY Stony Brook.

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[3] L. Block and J. Keesling: Computing the topological entropy of maps of the interval with three monotone pieces. Journal of Statistical Physics, **66**, 2 (1992).