40. *All Congruent Numbers less than 10000*

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1. Notations and result. If n is a square free positive integer such that
$$n = xy/2, \quad x^2 + y^2 = z^2$$
for some rational numbers x, y, z, then n is said to be a congruent number. If $x/y = u^2$ for some rational number u, we write $x \simeq y$. Because of the well-known expressions of Pythagorean numbers, it is clear that n is a congruent number if and only if

(1) $$n \simeq ab(a-b)(a+b)$$

for some positive integers a, b, $0 < b < a$. If

(2) $$n \simeq cd(c^2 + d^2)/2$$

for some positive integers c, d, $0 < d < c$, then
$$n \simeq (c+d)^2 (c-d)^2 \{(c+d)^2 - (c-d)^2\}\{(c+d)^2 + (c-d)^2\}.$$
Therefore n is a congruent number (cf. [3], [10]).

Now, let E_n be the elliptic curve
$$E_n : y^2 = x^3 - n^2 x.$$
It is known that n is a congruent number if and only if the rank $r(E_n)$ of E_n is positive (cf. [4], [7]). It was conjectured by Birch and Swinnerton-Dyer [1] that $L(E_n, 1) = 0$ is equivalent with $r(E_n) > 0$ where $L(E_n, s)$ is the Hasse-Weil L-function of E_n. It was proved, furthermore, by [5], that if $r(E_n) > 0$ then $L(E_n, 1) = 0$, and Tunnell's theorem [11] permits us easily to determine if $L(E_n, 1) = 0$ or not for given n. Using computer we got the next result.

Main result. Let n be a square free positive integer less than 10000, then
$$n = \text{a congruent number} \Longleftrightarrow L(E_n, 1) = 0.$$
More precisely, if $n \equiv 5,6,7 \pmod 8$ then n is a congruent number and if $n \equiv 1,2,3 \pmod 8$ then all such congruent numbers are listed in Table II.

2. Methods. If $n \equiv 5,6,7 \pmod 8$, then we have $L(E_n, 1) = 0$ (this can be shown without using [11]. Cf. [7]). In [6] it was proved that if $L(E_n, 1) = 0$, $L'(E_n, 1) \neq 0$ then $r(E_n) > 0$. We computed $L'(E_n, 1)$ in the range $n \equiv 5,6,7 \pmod 8$, $n < 10000$(cf. [2]). We got $L'(E_n, 1) \neq 0$ except for next 15 numbers: $n =$ 1254,2605,2774,3502,4199,4669,4895,6286, 6671,7230,7766,8005,9015,9430,9654. For these 15 numbers we got $L^{(3)}(E_n, 1) \neq 0$ (cf. [2]) and we could easily find the solutions of (1) in the range $0 < b < a \leq 1601$ (cf. [9]). Therefore if $n \equiv 5,6,7 \pmod 8$ and $n < 10000$, then n is a congruent number. There are 3050 such numbers.

Using [11] we got 453 numbers such that $n \equiv 1,2,3 \pmod 8$, $n < 10000$, $L(E_n, 1) = 0$. For these 453 numbers we can find solutions of

(1) or (2) in the range $1 \leq b < a \leq 30000$, $1 \leq d < c \leq 30000$ except for 56 numbers which appear in Table I (cf. [9]). Now there is another sufficient condition for $r(E_n) > 0$ given in [4]: If one of the following diophantine equations (3), (4) is solvable, then $r(E_n) > 0$.

(3) $$dX^4 + (4n^2/d)Y^4 = Z^2,\ d \mid 4n^2,\ d \neq 1,$$

(4) $$dX^4 - (n^2/d)Y^4 = \pm Z^2,\ d \mid n^2,\ d \neq 1,\ d \neq n.$$

We have obtained by computer a solution of (3) or (4) which is listed in Table I. Therefore we get the Main result.

Table I. Solutions of (3) or (4)

	n	d	X	Y		n	d	X	Y
(4)	2306	2	601	7	(3)	7001	2	1009	1
(4)	2818	2	849	17	(4)	7426	2	9921	127
(3)	2833	5666	15851	1205	(4)	7498	23	91	12
(3)	3409	2	721	1	(3)	7561	2	4001	31
(3)	3433	6866	395	967	(3)	7563	2521	107125	19548
(3)	3554	1777	227	108	(4)	7658	98	161	107
(3)	3761	7522	329	513	(3)	7793	15586	103031	30735
(3)	3899	557	889	79	(3)	7795	5	839	19
(3)	4747	101	274	5	(4)	7874	62	127	423
(3)	4793	4793	697	184	(3)	8002	4001	64499	123130
(3)	5449	2	233	1	(4)	8258	4129	1511	1020
(3)	5473	26	391	25	(3)	8571	11428	2054	2127
(3)	5569	2	217	1	(4)	8578	4289	2009	940
(3)	5657	11314	15	2059	(3)	8593	8593	137	252
(3)	5794	2897	233	19	(3)	8609	2	41	7
(4)	5962	2	737	7	(3)	8906	73	456	37
(3)	6058	233	131	749	(3)	8962	4481	95	212
(3)	6073	12146	9775	3269	(3)	9049	9049	233	432
(3)	6274	3137	221	404	(3)	9122	18244	52731	44410
(4)	6386	31	25	16	(3)	9131	397	29	57
(4)	6411	3	13	7	(4)	9226	7	531	16
(3)	6529	2	4153	31	(3)	9257	9257	352543	269204
(3)	6657	634	665	3	(4)	9347	719	607	240
(4)	6658	2	791	17	(3)	9379	113	583	122
(4)	6683	41	677	20	(3)	9497	18994	51071	16515
(3)	6865	10	129	13	(4)	9642	4821	9125	3421
(3)	6914	13828	4869	2602	(4)	9658	4829	7413	3865
(4)	6995	1399	2042	603	(3)	9986	19972	46537	84

Table Ⅱ. All congruent numbers such that $n \equiv 1,2,3 \pmod 8$, $n < 10000$

34	41	65	137	138	145	154	161	194	210
219	226	257	265	291	299	313	323	330	353
371	386	395	410	426	434	442	457	465	505
514	546	561	602	609	651	658	674	689	721
723	731	761	777	793	866	889	890	905	915
985	987	995	1003	1057	1073	1081	1105	1113	1122
1131	1145	1146	1154	1155	1169	1178	1185	1186	1195
1201	1217	1241	1249	1282	1321	1330	1339	1346	1379
1387	1393	1411	1419	1434	1443	1482	1513	1561	1595
1610	1633	1635	1649	1651	1659	1705	1731	1745	1762
1770	1785	1794	1858	1939	1995	2035	2059	2113	2130
2139	2145	2154	2170	2195	2201	2249	2257	2273	2282
2298	2306	2329	2337	2379	2418	2434	2465	2530	2569
2594	2611	2698	2706	2730	2777	2810	2818	2833	2849
2865	2914	2929	2945	2953	2995	3001	3003	3017	3018
3026	3081	3090	3099	3122	3129	3145	3161	3171	3265
3281	3289	3290	3355	3379	3409	3433	3434	3435	3443
3458	3554	3570	3593	3594	3595	3601	3603	3634	3674
3705	3713	3746	3761	3778	3794	3841	3842	3865	3881
3882	3899	3930	3939	3985	4002	4010	4017	4033	4121
4123	4154	4179	4218	4242	4249	4258	4290	4305	4370
4379	4402	4411	4441	4466	4481	4490	4529	4587	4595
4633	4634	4641	4649	4674	4681	4705	4731	4747	4763
4793	4794	4810	4834	4841	4843	4867	4889	4898	4899
4921	5002	5065	5066	5073	5105	5115	5138	5161	5185
5195	5226	5259	5273	5306	5314	5362	5394	5449	5465
5466	5467	5473	5474	5506	5545	5546	5569	5587	5593
5602	5610	5657	5666	5738	5747	5794	5809	5849	5865
5962	5986	6001	6010	6051	6058	6065	6073	6083	6090
6106	6242	6251	6274	6290	6305	6346	6355	6386	6402
6409	6411	6441	6529	6531	6545	6555	6594	6641	6649
6657	6658	6674	6681	6683	6690	6706	6745	6769	6771
6865	6882	6906	6914	6923	6953	6963	6995	7001	7170
7210	7289	7298	7321	7361	7394	7395	7410	7417	7426
7449	7473	7498	7561	7563	7579	7585	7602	7609	7611
7633	7658	7689	7707	7721	7747	7769	7779	7793	7795
7819	7874	7881	7995	8002	8090	8106	8122	8138	8162
8169	8170	8194	8234	8241	8258	8313	8354	8385	8393
8498	8515	8521	8547	8554	8555	8571	8578	8593	8601
8609	8690	8705	8729	8762	8786	8801	8809	8835	8859
8866	8890	8897	8906	8931	8962	8970	8979	9049	9066
9089	9121	9122	9131	9170	9226	9257	9281	9339	9345
9347	9354	9377	9379	9395	9401	9434	9435	9442	9451
9465	9483	9490	9497	9521	9546	9554	9579	9595	9601
9618	9634	9641	9642	9658	9690	9730	9731	9779	9809
9841	9867	9986							

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