38. A Remark to the Paper "On the Stabilizer of Companion Matrices" by J. Gomez-Calderon

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In the paper [1] cited in the title, the following question is treated.

Let R be a commutative ring with 1, M the ring of $n \times n$ matrices over R, $n \geq 2$, $f(X) = X^n - \sum_{i=0}^{n-1} b_i X^i$ a polynomial of degree n in $R[X]$, $C(f)$ the companion matrix of $f(X)$ defined by

$$C(f) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & b_0 \\ 1 & \ddots & & \vdots & b_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & b_{n-1} \end{pmatrix}$$

(which has $f(X)$ as the characteristic polynomial). An element A of M such that

$$(*) \qquad AC(f) = C(f)A$$

is called a stabilizer of $f(X)$, the set of which will be denoted by $S(f)$. In [1], a characterization of $A \in S(f)$ is given, and as an application, the following result is proved:

If R is a finite field F and $f(X)$ is irreducible, then every non-zero element of $S(f)$ is invertible.

The proof of this fact given in [1] is based on [2] and uses essentially the finiteness of F. In this note, another characterization of $S(f)$ will be given (Theorem 1) and it will be proved that the above proposition holds for any field F (Theorem 2).

The above notations R, M, $C(f)$, $S(f)$ will be used in the same meanings throughout this note.

Theorem 1. *$\boldsymbol{a}_1,\dots,\boldsymbol{a}_n$ being n column vectors $\in \boldsymbol{R}^n$, the follwing four conditions on $A = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_n) \in M$ are mutually equivalent.*

(1) $AC(f) = C(f)A, \quad i.e. \quad A \in S(f),$

(2) $\boldsymbol{a}_{i+1} = C(f)\boldsymbol{a}_i, \quad i = 1,2,\dots, n-1,$

(3) $\boldsymbol{a}_i = C(f)^{i-1}\boldsymbol{a}_1, \quad i = 1,2,\dots, n,$

(4) *A can be expressed as $g(C(f))$, $g(X) \in R[X]$, $deg\, g(X) \leq n-1$.*

Proof. In computing both sides of $(*)$ for $A = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$ we obtain

$$(\boldsymbol{a}_2,\dots, \boldsymbol{a}_n, \sum_{i=0}^{n-1} b_i \boldsymbol{a}_{i+1}) = (C(f)\boldsymbol{a}_1,\dots, C(f)\boldsymbol{a}_n).$$

Comparing the first $n-1$ columns, we see

$$\boldsymbol{a}_{i+1} = C(f)\boldsymbol{a}_i, \quad i = 1,2,\dots,n-1.$$

This coincides with (2), and we have (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is obvious.

Now suppose A satisfies (3) i.e.

$$A = (\boldsymbol{a}_1, C(f)\boldsymbol{a}_1, \ldots, C(f)^{n-1}\boldsymbol{a}_1).$$

Considering $f(X)$ as fixed, this matrix A is determined by $\boldsymbol{a}_1$. We shall therefore write $A(\boldsymbol{a}_1)$ for above A. Obviously $A(\boldsymbol{a})$ depends linearly on $\boldsymbol{a}: A(a\boldsymbol{a} + b\boldsymbol{b}) = aA(\boldsymbol{a}) + bA(\boldsymbol{b})$, $a, b \in \boldsymbol{R}$, $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^n$. Put now

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} a_i \boldsymbol{e}_i.$$

From the form of $C(f)$, one sees immediately

$$\boldsymbol{e}_{i+1} = C(f)\boldsymbol{e}_i = C(f)^i \boldsymbol{e}_1, \quad i = 1,2,\ldots, n-1$$

and

$$C(f)\boldsymbol{e}_n = \sum_{i=0}^{n-1} b_i \boldsymbol{e}_{i+1} = (\sum_{i=0}^{n-1} b_i C(f)^i)\boldsymbol{e}_1 = C(f)^n \boldsymbol{e}_1$$

where the last equality

$$\sum_{i=0}^{n-1} b_i C(f)^i = C(f)^n$$

follows from Hamilton-Cayley's theorem, as $f(X)$ is the characteristic polynomial of $C(f)$.

Thus we have

$$A(\boldsymbol{e}_1) = (\boldsymbol{e}_1, C(f)\boldsymbol{e}_1, \ldots, C(f)^{n-1}\boldsymbol{e}_1) = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = E,$$

$$A(\boldsymbol{e}_2) = (C(f)\boldsymbol{e}_1, C(f)^2\boldsymbol{e}_1, \ldots, C(f)^n\boldsymbol{e}_1) = C(f)A(\boldsymbol{e}_1) = C(f),$$

$$\cdots$$

$$A(\boldsymbol{e}_n) = C(f)^{n-1}$$

and so

$$A(\boldsymbol{a}) = a_1 E + a_2 C(f) + \ldots + a_n C(f)^{n-1}$$

which shows (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (1) is obvious.

Theorem 2. *Let R be a field F and $f(X)$ be irreducible in $F[X]$. Then any non-zero element A of $S(f)$ is invertible.*

Proof. By (4) of the last Theorem, $A \in S(f)$ can be written in the form $g(C(f))$ where $g(X) = \sum_{i=0}^{n-1} a_i X^i \in F[X]$ and $g(X) \neq 0$ as $A \neq 0$. The eigen values $\alpha_1, \ldots, \alpha_n$ of $C(f)$ are the roots of $f(X) = 0$, and as $f(X)$ is irreducible of degree n, $g(\alpha_i) \neq 0$, $i = 1, \ldots, n$. Now the eigen values of $A = g(C(f))$ are $g(\alpha_i)$, $i = 1, \ldots, n$ are $\det A = g(\alpha_1) \cdots g(\alpha_n) \neq 0$. So A is invertible.

Remark. The following is known. (Cf. e.g. [3].) Let C be an $n \times n$ matrix over a field F such that its characteristic polynomial coincides with its minimal polynomial. Then any matrix $A \in M$ which commutes with C can be expressed as $g(C)$, $g(X) \in F[X]$, $\deg g \leq n - 1$. But if we take a commutative ring R instead of a field F, this statement is not true in general.

References

[1] J. Gomez-Calderon: On the stabilizer of companion matrices. Proc. Japan Acad., **69** A, 140–143 (1993).
[2] L. E. Dickson: Linear Groups. New York (1958).
[3] K. Shoda: Ueber die mit einer Matrix vertauschbaren Matrizen. Math. Z., **29**, 696–712 (1929).