36. *A Note on the Rational Approximations to tanh* $\frac{1}{k}$

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§ 1. Introduction. I. Shiokawa [4] proved the following theorem.

Theorem A. *Let k be a positive integer. Then there is a positive constant C depending only on k such that*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>C\frac{\log\log q}{q^2\log q}$$

for all integers p and q with $q\geq 3$.

The purpose of this note is to prove the following theorem which shows that constant C in Theorem A is an effectively computable number depending only on $k\geq 2$.

Theorem. *Let k and N be positive integers with $k\geq 2$ and $N\geq 10$, and let p_n/q_n be the n-th convergent of $\tanh\frac{1}{k}$. Let γ_N and δ_n be defined by*

$$\gamma_N=2\left(k+\frac{k+1}{N-1/2}\right)\left(1+\frac{\log\log(2k(N+1)/e)}{\log(N+1)}\right)$$

and

$$\delta_n=\frac{(k(2n+1)+2)\log\log q_n}{\log q_n},$$

respectively. Let γ be any constant such that

$$\gamma\geq\max\{\gamma_N,\gamma_N^*\},$$

where

$$\gamma_N^*=\max\{\delta_n\mid 1\leq n<N\}.$$

Then

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\frac{\log\log q}{\gamma q^2\log q}$$

for all integers p and q with $q\geq 2$.

We now record two corollaries of the theorem.

Corollary 1. *For all integers p and q with $q\geq 2$,*

$$\left|\tanh\frac{1}{2}-\frac{p}{q}\right|>\frac{\log\log q}{6q^2\log q}.$$

Corollary 2. *For all integers p and q with $q\geq 2$,*

$$\left|\tanh\frac{1}{3}-\frac{p}{q}\right|>\frac{\log\log q}{9q^2\log q}.$$

§ 2. Lemma. Lemma. *Under the same assumptions as in Theorem,*

$$\left|\tanh\frac{1}{k}-\frac{p}{q}\right|>\frac{\log\log q}{\gamma_N q^2\log q}$$

for all integers p and q with $q \geq q_N$.

Proof. If p/q is not a convergent of $\tanh \frac{1}{k}$, then

$$\left| \tanh \frac{1}{k} - \frac{p}{q} \right| > \frac{1}{2q^2}.$$

Therefore, the lemma is proved in this case. We must consider the case that p/q is a convergent of $\tanh \frac{1}{k}$. The continued fraction of $\tanh \frac{1}{k}$ is

$$\tanh \frac{1}{k} = [a_0, a_1, a_2, a_3, \cdots] = [0, k, 3k, 5k, \cdots].$$

In other words, $a_0 = 0$ and $a_n = k(2n-1)$ for $n \geq 1$. Since $q_{n+1} = a_{n+1}q_n + q_{n-1} = k(2n+1)q_n + q_{n-1} < (k(2n+1)+1)q_n$, we have

$$\left| \tanh \frac{1}{k} - \frac{p_n}{q_n} \right| > \frac{1}{q_n(q_{n+1}+q_n)} > \frac{1}{(k(2n+1)+2)q_n^2}.$$

Now we must estimate q_n. Suppose that $n \geq N$. Since $q_n \geq k(2n-1)q_{n-1} \geq \cdots \geq k^n \Pi_{\nu=1}^{n}(2\nu-1)$, we have

$$\begin{aligned}
\log q_n &\geq n \log k + \sum_{\nu=1}^{n} \log(2\nu-1) \\
&\geq n \log k + \int_1^n \log(2x-1)dx \\
&= n \log k + (n-1/2)\log(2n-1) - n + 1 \\
&\geq (n-1/2)\log((2n-1)/e^{1/3}).
\end{aligned}$$

Conversely, $q_n \leq 2knq_{n-1}$. Hence,

$$q_n \leq (2k)^n n!.$$

Therefore,

$$\begin{aligned}
\log q_n &\leq n \log(2k) + \sum_{\nu=1}^{n} \log \nu \\
&\leq n \log(2k) + \int_1^{n+1} \log x dx \\
&= n \log(2k) + (n+1)\log(n+1) - n \\
&\leq (n+1)\log(2k(n+1)/e), \\
\log\log q_n &\leq \log(n+1) + \log\log(2k(n+1)/e).
\end{aligned}$$

As we can see that

$$l(x) = \frac{\log\log(2k(x+1)/e)}{\log(x+1)} \quad (x \geq 10)$$

is a strictly decreasing function, we have

$$\log\log q_n \leq (1+l(N))\log(n+1) \leq (1+l(N))\log((2n-1)/e^{1/3}).$$

From these consequences, we find

$$\begin{aligned}
\frac{\log\log q_n}{\log q_n} &\leq \frac{1+l(N)}{n-1/2} \\
&\leq 2\left(k + \frac{k+1}{N-1/2}\right)\left(1 + \frac{\log\log(2k(N+1)/e)}{\log(N+1)}\right) \cdot \frac{1}{k(2n+1)+2} \\
&= \frac{\gamma_N}{k(2n+1)+2}.
\end{aligned}$$

Therefore,

$$\left| \tanh \frac{1}{k} - \frac{p_n}{q_n} \right| > \frac{\log \log q_n}{\gamma_N q_n^2 \log q_n}.$$

This completes the proof.

§ 3. Proof of the theorem. It suffices only to consider that p/q is an n-th convergent of $\tanh \frac{1}{k}$. From the definition of γ_N^*, we have following inequalities

$$\left| \tanh \frac{1}{k} - \frac{p_n}{q_n} \right| > \frac{1}{(k(2n+1)+2)q_n^2} = \frac{\log \log q_n}{\delta_n q_n^2 \log q_n} \geq \frac{\log \log q_n}{\gamma_N^* q_n^2 \log q_n} \quad (1 \leq n < N).$$

And from Lemma, we have

$$\left| \tanh \frac{1}{k} - \frac{p_n}{q_n} \right| > \frac{\log \log q_n}{\gamma_N q_n^2 \log q_n} \ (n \geq N).$$

This completes the proof of the theorem.

§ 4. Proof of corollaries. *Proof of Corollary* 1. For $N = 22$, we have $\gamma_{22} = 5.9972\cdots$ and $\gamma_{22}^* = \delta_5 = 5.3972\cdots$. Hence we can choose γ so that $\gamma = 6$. Then Corollary 1 follows at once from the theorem.

Proof of Corollary 2. For $N = 27$, we have $\gamma_{27} = 8.9813\cdots$ and $\gamma_{27}^* = \delta_8 = 7.1487\cdots$. Hence we can choose γ so that $\gamma = 9$. Then Corollary 2 follows at once from the theorem.

References

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