2. *Curves and Symmetric Spaces*

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This is an announcement of our research on the classification of curves, i.e., compact Riemann surfaces, of genus $g=7$, 8 and 9 and their canonical rings by means of the symmetric spaces $X_{2g-2}^{24-2g}\subset \boldsymbol{P}^{22-g}$ studied in [3]. The details will be published elsewhere. A line bundle L on a curve C is a g_d^r if $\deg L=d$ and $\dim H^0(C,L)\geq r+1$.

§ 1. **Linear section theorems.** A non-hyperelliptic curve C embedded in $\boldsymbol{P}^{g-1}$ by the canonical linear system $|K_C|$ is called a *canonical curve*. The canonical ring of C is isomorphic to the homogeneous coordinate ring of $C\subset \boldsymbol{P}^{g-1}$ by Noether's theorem.

Let $X_{12}^8\subset \boldsymbol{P}^{14}$ be the 8-dimensional complex Grassmannian $U(6)/(U(2)\times U(4))$ embedded in $\boldsymbol{P}^{14}$ by the Plücker coordinates. It is classically known that a transversal linear subspace P of dimension 6 cut out a curve C of genus 8 and that the embedding $C\subset P$ is canonical.

Theorem 1. *A curve C of genus* 8 *is a transversal linear section of the* 8-*dimensional Grassmannian if and only if C has no g_7^2.*

Complex Grassmannians are symmetric spaces of type AIII. Besides $X_{14}^8\subset \boldsymbol{P}^{14}$ two compact Hermitian symmetric spaces $X_{12}^{10}\subset \boldsymbol{P}^{15}$ and $X_{16}^6\subset \boldsymbol{P}^{13}$ yield canonical curves (of genus 7 and 9) as transversal linear sections. The former is $SO(10)/U(5)$ of type DIII embedded in the projectivization of the space U^{16} of semi-spinors. Let $\mathrm{Alt}_5\,\boldsymbol{C}$ be the space of skew-symmetric matrices of degree 5. Then $X_{12}^{10}\subset \boldsymbol{P}^{15}$ is the compactification of the embedding

$$\begin{array}{ccc}\mathrm{Alt}_5\,\boldsymbol{C} & \longrightarrow & \boldsymbol{P}^{15},\\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$}\\ A=(a_{ij}) & \longmapsto & (1:a_{12}:\cdots:a_{45}:\mathrm{Pfaff}\,A^1:\cdots:\mathrm{Pfaff}\,A^5)\end{array}$$

where $A^1,\cdots,A^5$ are the principal minors of A. The latter is the compact dual $Sp(3)/U(3)$ of the Siegel upper half space $\mathfrak{H}_3$ of degree 3 embedded in the projectivization of a 14-dimensional irreducible representation U^{14} of $Sp(3)$. Let $\mathrm{Sym}_3\,\boldsymbol{C}$ be the space of symmetric matrices of degree 3. Then $X_{16}^6\subset \boldsymbol{P}^{13}$ is the compactification of the Veronese-like embedding

$$\begin{array}{ccc}\mathrm{Sym}_3\,\boldsymbol{C} & \longrightarrow & \boldsymbol{P}(\boldsymbol{C}\oplus \mathrm{Sym}_3\,\boldsymbol{C}\oplus \mathrm{Sym}_3\,\boldsymbol{C}\oplus \boldsymbol{C}),\\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$}\\ A & \longmapsto & (1:A:A':\det A)\end{array}$$

where A' is the cofactor matrix of A.

Theorem 2. *A curve C of genus* 7 (resp. 9) *is a transversal linear section of $X_{12}^{10}\subset \boldsymbol{P}^{15}$* (resp. $X_{16}^6\subset \boldsymbol{P}^{13}$) *if and only if C has no g_5^1* (resp. g_6^1).

Example. The symmetric space $X_{12}^{10} \subset \boldsymbol{P}^{15}$ has a faithful action of the finite simple group $SL(2, \boldsymbol{F}_8)$ of order 504 and has two invariant subspaces P_1 and P_2 of dimension 6 and 8, respectively. The intersection $C = P_1 \cap X_{12}^{10}$ is a curve of genus 7 which satisfies $|\mathrm{Aut}\, C| = 84(g-1)$. This curve is constructed from a quaternion algebra over $\boldsymbol{Q}(\cos 2\pi/7)$ (cf. [5] Remark 3.19). The other intersection $P_2 \cap X_{12}^{10}$ is a Fano 3-fold of genus 7 with Picard number one.

Remark. (1) The representations U^{16} of $Spin(10)$ and U^{14} of $Sp(3)$ are studied in [2]. Both $\boldsymbol{P}(U^{16})$ and $\boldsymbol{P}(U^{14})$ have open dense orbits.

(2) A curve C of genus g is *numerically Petri general* if $h^0(L)h^0(\omega_C L^{-1}) \leqq g$ holds for every line bundle L on C. In the case $g=7$ (rep. 8, 9), C is numerically Petri general if and only if it has no g_4^1 (rep. g_7^2, g_5^1).

§ 2. Birational type of M_g. Let $C \subset \boldsymbol{P}^6$ be a canonical curve of genus 7 and $N_{C/P}^*$ its conormal bundle. We denote the space of quadratic forms on $\boldsymbol{P}^6$ vanishing on C by V and that of quartic forms vanishing doubly along C by W. By the Enriques-Petri theorem ([4]), the rank 5 vector bundle $N_{C/P}^* \otimes O_C(2K)$ is generated by V if C has no g_3^1. Since $\dim V = 10$, the pair $(V, N_{C/P}^* \otimes O_C(2K))$ defines a morphism ψ of C to the 25-dimensional Grassmannian $G(5, V)$. The embedding of C into X_{12}^{10} in Theorem 2 is constructed as follows:

Proposition 1. *Let C be a curve of genus 7 without g_4^1. Then we have*

(1) *$\psi: C \to G(5, V)$ is an embedding, and*

(2) *the natural map $S^2V \to W$ is not injective and its kernel is generated by a non-degenerate symmetric tensor.*

By (2) of the proposition, the image of ψ is contained in a symmetric space X_{12}^{10}.

In the case $g=8$, 9, we construct special vector bundles in order to embed C into the corresponding symmetric spaces. A vector bundle is *quasi-stable* if it is a direct sum of stable vector bundles with the same slope.

Definition. $E_C(r, K)$ is the set of isomorphism classes of rank r quasi-stable vector bundles E on C with canonical determinant, i.e., $\Lambda^r E \simeq O_C(K)$. $\eta_r(C)$ is the maximum of $\dim H^0(C, E)$ when E runs over $E_C(r, K)$.

Proposition 2. (1) *If C is a curve of genus 8 and has no g_4^1, then $\eta_2(C) = 6$ and the maximum is attained by the unique 2-bundle $E_{\max} \in E_C(2, K)$.*

(2) *If C is a curve of genus 9 and has no g_5^1, then $\eta_3(C) = 6$ and the maximum is attained by the unique 3-bundle $E_{\max} \in E_C(3, K)$.*

The embeddings of C into X_{14}^8 and X_{16}^6 in Theorems 1 and 2 are constructed from the complete linear system associated to $E_{\max}$. Hence the embeddings are strongly rigid:

Theorem 3. *Assume that two linear subspaces P_1 and P_2 cut out curves C_1 and C_2 from the symmetric space $X_{2g-2}^{24-2g} \subset \boldsymbol{P}^{22-g}$ ($g=7$, 8 or 9),* *respectively. Then any isomorphism from* C_1 *onto* C_2 *extends to an automorphism* ϕ *of* $X^{24-2g}_{2g-2}\subset \boldsymbol{P}^{22-g}$ *with* $\phi(P_1)=P_2$.

Let M_g be the moduli space of curves of genus g. By the theorem, we have the following:

g	7	8	9
dim M_g	18	21	24
Birational type of M_g	$G(7, U^{16})/Spin(10)$	$G(8, U^{15})/SL(6)$	$G(9, U^{14})/Sp(3, \boldsymbol{C})$

§ 3. **Syzygies of canonical rings.** Let R_X be the homogeneous coordinate ring of the symmetric space $X^6_{16}\subset \boldsymbol{P}^{13}$. R_X is generated by 14 linear forms and the relation ideal is generated by 21 quadratic relations. Let S be the polynomial ring of 14 variables. As an S-module, R_X has the following minimal free resolution:

$$\begin{array}{l} 0\longleftarrow R_X\longleftarrow S\longleftarrow S(-2)\otimes U^{21}\longleftarrow S(-3)\otimes U^{64}\longleftarrow S(-4)\otimes(U^6\oplus U^{64}) \\ \hspace{27em}\uparrow \\ 0\longrightarrow S(-10)\longrightarrow S(-8)\otimes U^{21}\longrightarrow S(-7)\otimes U^{64}\longrightarrow S(-6)\otimes(U^6\oplus U^{64}) \end{array}$$

where U^i denotes an i-dimensional irreducible representation of $Sp(3)$. If a curve C is a transversal linear section of $X^6_{16}\subset \boldsymbol{P}^{13}$, then its canonical ring

$$R_C=\bigoplus_{n\geq 0} H^0(C, O_C(nK))$$

has the same type of resolution as a module over the polynomial ring of 9 variables. Hence Theorem 2 answers Green's conjecture ([1]) affirmatively in the case genus 9 since non-existence of g^1_5 is equivalent to Cliff $C=4$.

Theorem 4. *A canonical curve* $C\subset \boldsymbol{P}^8$ *of genus* 9 *satisfies Green's property* (N_p) *if and only if* Cliff $C>p$.

§ 4. **Canonical curves of genus 7 and 8.** Let $C\subset \boldsymbol{P}^{g-1}$ be a canonical curve of genus g.

Proposition 3. *If* C *has a* g^2_6*, then we have one of the following*:

a) $C\subset \boldsymbol{P}^{g-1}$ *is a hyperquadric section of a normal surface* $S\subset \boldsymbol{P}^{g-1}$ *of degree* $g-1$*, or*

b) C *has a* g^1_3*, or*

c) C *is a smooth plane quintic.*

In the case a), such surfaces are classified by del Pezzo. S is either the anticanonical model of a rational surface or the cone over an elliptic curve. In the case b) or c), the quadric hull of $C\subset \boldsymbol{P}^{g-1}$ is a surface of degree $g-2$ (cf. [4]).

If $g\neq 10$, every curve with a g^2_6 has a g^1_4. If $g=8$, every curve with a g^1_4 has a g^2_7. The classification of curves of genus 7 and 8 is completed by the following two propositions:

Proposition 4. *Let* C *be a curve of genus* 7. *If* C *has a* g^1_4 *but no* g^2_6*, then* C *is the complete intersection of three divisors of bidegree* $(1,1)$, $(1,2)$ *and* $(1,2)$ *in* $\boldsymbol{P}^1\times \boldsymbol{P}^3$.

Proposition 5. *Let C be a curve of genus* 8.

(1) *If C has a g^1_4 but no g^2_6, then C is the complete intersection of four divisors of bidegree* $(1,1)$, $(1,1)$, $(0,2)$ *and* $(1,2)$ *in* $\boldsymbol{P}^1 \times \boldsymbol{P}^4$.

(2) *If C has a g^2_7 but no g^1_4, then we have either*

a) *C is the complete intersection of three divisors of bidegree* $(1,1)$, $(1,2)$ *and* $(2,1)$ *in* $\boldsymbol{P}^2 \times \boldsymbol{P}^2$, *or*

b) *C is the common zero locus of the five* 4×4 *Pfaffians of a skew-symmetric matrix*

$$\begin{bmatrix} 0 & a_1 & a_2 & b_1 & b_2 \\ -a_1 & 0 & a_3 & b_3 & b_4 \\ -a_2 & -a_3 & 0 & b_5 & b_6 \\ -b_1 & -b_3 & -b_5 & 0 & c \\ -b_2 & -b_4 & -b_6 & -c & 0 \end{bmatrix}$$

in the weighted projective space $\boldsymbol{P}(1:1:1:2:2)$, *where* a_1, a_2, a_3 *are linear forms,* $b_1, \cdots, b_6$ *quadratic and c is a cubic form.*

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