15. *A Note on a Deformation of Dirichlet's Class Number Formula*

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§0. **Introduction.** In [3], Prof. T. Ono obtained interesting results from a deformation of Dirichlet's class number formula for real quadratic fields $Q(\sqrt{p})$, where p is a prime number of the form $p=4N+1$. In [2], the author gave a similar deformation in the case where p is a prime number of the form $p=4N+3$.

After the completion of [2], the author found that Dirichlet had already given a deformation of the class number formula for binary quadratic forms ([1], §107–§110 and §138–§140), which is, however, somewhat complicated. The purpose of this note is to give a more simple formula for any real quadratic field using the same methods as [2] and [3]. To be more precise, let m be a square-free positive integer, ε the fundamental unit >1 of the real quadratic field $Q(\sqrt{m})$ and h the class number of $Q(\sqrt{m})$. We denote by d the discriminant of $Q(\sqrt{m})$. The discriminant d is written in the form $d=P\ (\equiv 1 \bmod 4)$, $4P$ or $8P$, where $P=1$ or $P=p_1p_2\cdots p_r(p_1, p_2, \cdots, p_r$ are distinct odd prime numbers). ζ denotes a primitive dth root of unity. Let χ be a Kronecker character belonging to $Q(\sqrt{m})$, and $L(s,\chi)$ the corresponding L-series. As usual, we denote by ϕ the Euler function, and by μ the Möbius function. For the sake of simplicity, we denote $\phi(d)/4$ by v. For any integer $1\leqq t\leqq v$, define τ_t by putting

$$\tau_t=((\phi(d)/\phi(d/n))\cdot\mu(d/n)-\chi(t)\sqrt{d})/2, \qquad \text{where } n=(t,d).$$

We also define W as follows

$$W=\begin{cases}0, & \text{if } d \text{ has at least two distinct prime factors,}\\ 1, & \text{otherwise.}\end{cases}$$

Then our main theorem reads.

Theorem. *With the above notations, we have*

$$\sqrt{m^W}\varepsilon^h=2\sum_{j=0}^{v-1}d_j+d_v.$$

Here d_j are determined by the following recurrence relation.

$$jd_j=\sum_{t=1}^{j}\alpha_t d_{j-t} \quad (d_0=1,\ 1\leqq j\leqq v),$$

where $\alpha_t=-\tau_t$.

§1. **Dirichlet's formula.** It is known that (cf. [4])

$$h\kappa=L(1,\chi), \quad \text{where } \kappa=\frac{\log\varepsilon^2}{\sqrt{d}} \quad \text{and} \quad L(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}$$

and

$$\sum_{r \bmod d}\chi(r)\zeta^{nr}=\chi(n)\sqrt{d} \qquad \text{(the Gauss sum).}$$

Therefore

$$\log \varepsilon^{2h} = \sum_{n=1}^{\infty} \frac{1}{n} \Big(\sum_{r \bmod d} \chi(r)\zeta^{nr} \Big)$$

$$= \sum_{r=0}^{d-1} \chi(r) \sum_{n=1}^{\infty} \frac{1}{n} \zeta^{nr} = -\sum_{r=0}^{d-1} \chi(r) \log(1-\zeta^r).$$

Then we consider two sets of integers

$A = \{a \in Z\,;\, 1 \leqq a \leqq d-1, \chi(a) = +1\}$, $\qquad B = \{b \in Z\,;\, 1 \leqq b \leqq d-1, \chi(b) = -1\}$,

and put

$$A(x) = \prod_{a \in A} (1 - x\zeta^a), \qquad B(x) = \prod_{b \in B} (1 - x\zeta^b).$$

It is easy to see

(1) $$\varepsilon^{2h} = B(1)/A(1).$$

$\Phi_n(x)$ denotes the nth cyclotonic polynomial, that is,

$\Phi_n(x) = \prod_{0<i<n, (i,n)=1} (x - \zeta_n^i)$, where ζ_n is a primitive nth root of unity.

Then it is well-known that

$$\Phi_n(x) = \prod_{z|n} (x^z - 1)^{\mu(n/z)},$$

and one can easily show that

(2) $$\Phi_d(1) = A(1)B(1) = m^W \quad \text{(cf. [5])}.$$

From (1) and (2), one gets

(3) $$\sqrt{m^W}\,\varepsilon^h = B(1).$$

For the proof of our theorem, we have to determine the coefficients of $B(x)$.

Lemma 1 (Newton's formula, cf. [1] § 139, [3] § 2). *For any complex numbers a_i $(1 \leqq i \leqq M)$, we put*

$$F(t) = \prod_{i=1}^{M} (1 - ta_i) = \sum_{j=0}^{M} b_j t^j.$$

Then

$$b_j = \frac{F^{(j)}(0)}{j\,!}, \quad \textit{and} \quad jb_j = -\sum_{t=1}^{j} b_{j-t} S_t \quad (1 \leqq j \leqq M), \quad \textit{where } S_t = \sum_{i=1}^{M} a_i^t.$$

Put

$$A(x) = \sum_{j=0}^{2v} c_j x^j, \qquad B(x) = \sum_{j=0}^{2v} d_j x^j.$$

Applying Lemma 1 to polynomials $A(x)$, $B(x)$, we get

$$jc_j = -\sum_{t=1}^{j} c_{j-t}\sigma_t, \quad \text{where } \sigma_t = \sum_{a \in A} \zeta^{at}, \quad (1 \leqq j \leqq 2v),$$

$$jd_j = -\sum_{t=1}^{j} d_{j-t}\tau_t, \quad \text{where } \tau_t = \sum_{b \in B} \zeta^{bt}, \quad (1 \leqq j \leqq 2v).$$

Lemma 2. *$A(x)$, $B(x)$ are reciprocal polynomials, i.e.,*

$$a_u = a_{2v-u}, \quad b_u = b_{2v-u}, \quad 0 \leqq u \leqq 2v \quad \text{(cf. [1])}.$$

Then, from Lemmas 1–2 and the equation (3) one gets

(4) $$\sqrt{m^W}\,\varepsilon^h = B(1) = \sum_{j=0}^{2v} d_j = 2\sum_{j=0}^{v-1} d_j + d_v.$$

Thus, to finish the proof of our theorem, we have only to determine the values of τ_t.

§ 2. The determination of τ_t. Next lemma is easily shown by the

Möbius's inversion formula (cf. [4] § 9, problem 2).

Lemma 3. *Let $f(n)$ be the sum of all the roots of $\Phi_n(x)=0$, then $f(n)=\mu(n)$.*

In the following we consider two cases (i) $(t, d)=1$ and (ii) $(t, d)\neq 1$.

(i) By Lemma 3, we get

$$\sigma_t+\tau_t=\sum_{x\in(Z/dZ)^\times}\zeta^{xt}=f(d)=\mu(d).$$

On the other hand, it is easy to see that

$$\sigma_t-\tau_t=\sum_{x\in(Z/dZ)^\times}\chi(x)\zeta^{xt}=\chi(t)\sqrt{d}\quad\text{(the Gauss sum).}$$

Hence, we have

$$\tau_t=(\mu(d)-\chi(t)\sqrt{d})/2. \tag{5}$$

(ii) In this case, it is easy to see $\sigma_t-\tau_t=\chi(t)\sqrt{d}=0$. We put $n=(t, d)$. Then ζ^{na} and ζ^{nb} are primitive (d/n)th roots of unity, each of which appears $\phi(d)/\phi(d/n)$ times. Then, by Lemma 3 we see that

$$\sigma_t+\tau_t=(\phi(d)/\phi(d/n))\cdot\mu(d/n).$$

Hence one gets

$$\tau_t=(\phi(d)/\phi(d/n))\cdot\mu(d/n)/2. \tag{6}$$

By unifying (5) and (6), we have

$$\tau_t=((\phi(d)/\phi(d/n))\cdot\mu(d/n)-\chi(t)\sqrt{d})/2.$$

Hence we have shown our main theorem in § 0.

§ 3. Some illustrations. (1) The case $d=33$ $(m=33)$. Then one has $\phi(33)=20$, $v=5$, $W=0$.

Put $\omega=(1+\sqrt{33})/2$, then $\omega^2=\omega+8$.

For $t=1$ to 5, $\chi(t)$, α_t and d_t are as follows.

t	0	1	2	3	4	5
$\chi(t)$		$+1$	$+1$	0	$+1$	-1
α_t		$\omega-1$	$\omega-1$	1	$\omega-1$	$-\omega$
d_t	1	$\omega-1$	4	$\omega+2$	2ω	7

Hence $\varepsilon^h=2(d_0+d_1+d_2+d_3+d_4)+d_5=23+4\sqrt{33}$.

On the other hand, $\varepsilon=23+4\sqrt{33}$. Hence $h=1$.

(2) The case $d=60$ $(m=15)$. Then one has ϕ $(60)=16$, $v=4$, $W=0$.

Put $\omega=\sqrt{15}$, then $\chi(t)$, α_t and d_t are as follows.

t	0	1	2	3	4
$\chi(t)$		$+1$	0	0	0
α_t		ω	1	0	-1
d_t	1	ω	8	3ω	13

Hence $\varepsilon^h=2(d_0+d_1+d_2+d_3)+d_4=31+8\sqrt{15}$.

On the other hand, $\varepsilon=4+\sqrt{15}$. Hence $h=2$.

(3) The case $d=8$ $(m=2)$. Then one has $\phi(8)=4$, $v=1$, $W=1$, $\alpha_1=\sqrt{2}$, and $d_1=\sqrt{2}$. Hence $\sqrt{2}\,\varepsilon^h=2+\sqrt{2}$. On the other hand $\varepsilon=1+\sqrt{2}$. Hence $h=1$.

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References

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