14. Global Solvability and Hypoellipticity on the Torus for a Class of Differential Operators with Variable Coefficients

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1. Notations and results. The purpose of the present note is to give conditions for global solvability and hypoellipticity for a class of second order differential operators on the two dimensional torus T^2. The principal result is a necessary and sufficient condition for the global solvability in terms of small denominator type estimates, a phenomena known so far only for differential operators with constant coefficients. We recall the well known result of S. Greenfield and N. Wallach [5] showing that the operator D_x+cD_y, $c\in R\backslash 0$ is globally hypoelliptic on T^2 if and only if c is an irrational non Liouville number, despite that it is always locally non-hypoelliptic. J. Hounie [6] proved a necessary and sufficient condition for global solvability for first order systems $\partial_t+b(t)A$, where A is an essentially self-adjoint operator, while D. Fujiwara and H. Omori [2] established global hypoellipticity for $D_x^2+\varphi(x)D_y^2$, φ being $C^\infty(R)$ 2π periodic real-valued function, identically equal to 0 and 1 on some subintervals of $[0, 2\pi]$. Recently, the third author studied global hypoellipticity of a Mathieu operator on T^2.

The present paper examines second order differential operators on the two dimensional torus $T^2=R^2/2\pi Z^2$ of the following type

$$P=(D_x+ia(x)D_y)(D_x+ib(x)D_y)+\gamma(x)D_y+c(x), \tag{1}$$

where $\gamma(x)$ equals either 0 or $(a'(x)-b'(x))$, $D_z=i^{-1}\partial_z$, $z=x$ or y, and where $a(x)$, $b(x)$, $c(x)\in C^\infty(T)$, i.e., 2π periodic C^∞ complex-valued functions on R. We will study the following equation in the space of periodic distributions $\mathcal{D}'(T^2)$

$$Pu=f, \qquad f\in\mathcal{D}'(T^2). \tag{2}$$

When $c\equiv 0$, the operator P has a remarkable property that the solutions to the homogeneous equation $Pu=0$ on T^2 are written explicitly.

We say that P is globally solvable (resp. hypoelliptic) if for every $f\in C^\infty(T^2)$ there exists a $u\in\mathcal{D}'(T^2)$ satisfying (2) (resp. $u\in C^\infty(T^2)$ when $Pu\in C^\infty(T^2)$ and $u\in D'(T^2)$). Similarly, P is said to be globally hypoelliptic in the Gevrey class $G^s(T^2)$ if $Pu\in G^s(T^2)$ and $u\in G^{s\prime}(T^2)$ implies $u\in G^s(T^2)$. P is said to be locally solvable (resp. hypoelliptic) at a point p if there

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exists a neighborhood U of p such that for any $f \in C_0^\infty(U)$, there exists $u \in \mathcal{D}'(U)$ such that $Pu=f$ in U (resp. $p \notin sing\ supp\ (Pu)$ implies $p \notin sing\ supp\ (u)$). Here $sing\ supp(u)$ stands for the singular support of u, namely the smallest closed subset of T^2 outside which u coincides with a C^∞ function. We point out that even in the case of local solvability and hypoellipticity for P few results are known for general $a(x)$ and $b(x)$ (for more details see [7] and references there).

We set $\omega_a = \int_0^{2\pi} Rea(x)dx$ and we define ω_b similarly. If $Rea(x)$ does not change its sign we denote by k_a the maximal order of vanishing of $Rea(x)$ with the convention that $k_a = +\infty$ (resp. $k_a = 0$) if $Rea(x)$ has a zero of infinite order (resp. $Rea(x)$ is nonzero at every point of T). In the same way we define k_b. Then we have

Theorem 1. *Assume that $Rea(x)$ and $Reb(x)$ do not change sign, both of them are not identically zero. Moreover, suppose that one of the following conditions holds; either*

(3) $\delta := \frac{1}{k_a+1} + \frac{1}{k_b+1} > 0$, *i.e. either $Rea(x)$ or $Reb(x)$ has only finite order zeros*

or

$$\sup|c(x)| < 1/q = \frac{(e^{|\omega_a|}-1)(e^{|\omega_b|}-1)}{12\pi^2 e^{|\omega_a|+|\omega_b|}}. \tag{4}$$

Then we have:

i) $\dim Ker(P) < \infty$ *and* $\dim Ker(P^*) < \infty$.

ii) *If $f \in \mathcal{D}'(T^2)$ the equation* (2) *has a solution if and only if $\langle f, \varphi \rangle_{\mathcal{D}'(T^2)} = 0$ for every $\varphi \in Ker(P^*)$.*

iii) *The operator P is globally hypoelliptic. Moreover, if $f \in H^s(T^2)$, $s \in R$ and if $u \in \mathcal{D}'(T^2)$ satisfies $Pu=f$ then $u \in H^{s+\delta}(T^2)$, i.e. the loss of smoothness for P is not greater than $2-\delta$. In the case when either k_a or k_b is finite, the loss of smoothness for the operator P_0 is exactly equal to δ.*

In particular, if $c(x) \equiv \lambda$ is a complex constant satisfying (4) *and $\lambda \neq -n^2$ for $n \in Z$, then the operator P is globally solvable and globally hypoelliptic.*

We stress that there are no restriction on the imaginary parts of $a(x)$ and $b(x)$, and in particular P may change its type, namely hyperbolic in some regions of T^2 (and thus it is not locally hypoelliptic) and elliptic in other regions. The result above contains novelty even in the local theory of solvability for operators with double characteristics.

Remark. If $Rea(x)$ or $Reb(x)$ changes its sign at some points we can prove local nonsolvability under certain additional restrictions (which implies global nonsolvability) and the existence of $u \in \mathcal{D}'(T^2)$ such that $Pu=0$ and $sing\ supp(u) \neq \emptyset$.

We show now that the result for global hypoellipticity in the previous theorem is stable under semilinear perturbations. More precisely, let $g(x,y,z)\in C^\infty(\boldsymbol{T}^2\times\boldsymbol{C})$ and assume in addition that g is an entire function with respect to z.

Theorem 2. *Let the assumptions in Theorem* 1 *be true and suppose, in addition, that* $\delta>0$. *Then if* $u(x,y)\in H^s(\boldsymbol{T}^2)$, $s>1$ *and* $Pu+g(x,y,u)\in C^\infty(\boldsymbol{T}^2)$, *it follows that* $u\in C^\infty(\boldsymbol{T}^2)$ *as well.*

Next we drop the requirement that the real parts of $a(x)$ and $b(x)$ do not vanish identically and propose, for some operators with variable coefficients, an analogue of the well known small divisor condition for operators with constant coefficients [5]. Put $\tau_a=\int_0^{2\pi}\mathrm{Im}\, a(x)dx$ and we similary define τ_b.

Theorem 3. *Suppose that* $Rea(x)$ *and* $Reb(x)$ *do not change their sign. Then if* $Rea(x)\equiv 0$ (*resp.* $Reb(x)\equiv 0$) *the equation*

(5) $$P_0u=(D_x+ia(x)D_y)(D_x+ib(x)D_y)u+(a'(x)-b'(x))D_yu=f$$

has a solution $u\in\mathcal{D}'(\boldsymbol{T}^2)$ *for every* $f\in C^\infty(\boldsymbol{T}^2)$ *such that* $\int_0^{2\pi}\int_0^{2\pi}f(x,y)dxdy=0$ *if and only if*

(6) $\tau_a/(2\pi)$ (*resp.* $\tau_b/(2\pi)$) *is an irrational non Liouville number.*

Moreover (6) *implies the global hypoellipticity of* P_0. *Assume now that the functions* $a(x)$ *and* $b(x)$ *are real-analytic. Then if* $Rea\equiv Reb\equiv 0$ *and* $\tau_a/(2\pi)$ (*resp.* $\tau_b/(2\pi)$) *is an irrational Liouville number and there exists* $s>1$ *having the property, for any* $0<\varepsilon\ll 1$, *there exists* $C_\varepsilon>0$ *such that*

$$|\tau-p/q|\geq C_\varepsilon e^{-\varepsilon q^{1/s}},\quad p\in\boldsymbol{Z},\ q\in\boldsymbol{Z}\backslash 0,$$

then P_0 *is globally hypoelliptic in the Gevrey class* $G^s(\boldsymbol{T}^2)$.

2. Sketch of the proof of the results. We first prove Theorems 1 and 3. For the sake of simplicity, we shall consider the operator P_0 (i.e. $c(x)=0$) in the most difficult case $\delta=0$. The general case will be proved by use of a WKB method. (Cf. [4]). We note that we cannot expect to gain the derivatives in case $\delta=0$. (For the case $\delta>0$ we also refer to the lemma which follows.)

Using the discrete Fourier transform $\hat{u}(x,\eta)=\int_0^{2\pi}e^{-i\eta y}u(x,y)dy$, $\eta\in\boldsymbol{Z}$ we reduce the equation (2) to

(7) $$\hat{P}_0\hat{u}(x,\eta)=((D_x+ia(x)\eta)(D_y+ib(x)\eta)+(a'(x)-b'(x))\eta)\hat{u}=\hat{f}(x,\eta).$$

We will solve (7) for $\eta\gg 1$ since the case $\eta\ll -1$ is treated similary. We assume without loss of generality that Rea and Reb are nonnegative. Then we set $A(x)=\int_0^x a(s)ds$ and $B(x)=\int_0^x b(s)ds$. Evidently ReA and ReB are nondecreasing and $ReA(2\pi)>0$, $ReB(2\pi)>0$. We can show that the unique 2π periodic solution with respect to x of (7), $\eta\in\boldsymbol{N}$ can be written as

(8) $$\hat{u}(x,\eta)=Q\hat{f}(x,\eta)=\sum_{j=1}^4 Q_j\hat{f}(x,\eta)$$

with

$$Q_1\hat{f}(x,\eta)=-\int_x^{2\pi}\left(\int_x^z e^{\eta\psi(t,x,z)}dt\right)\hat{f}(z,\eta)dz$$

$$Q_2\hat{f}(x,\eta)=-(e^{\eta A(2\pi)}-1)^{-1}\int_0^{2\pi}\left(\int_0^z e^{\eta\psi(t,x,z)}dt\right)\hat{f}(z,\eta)dz$$

$$Q_3\hat{f}(x,\eta)=-(e^{\eta B(2\pi)}-1)^{-1}\int_0^{2\pi}\left(\int_x^{2\pi} e^{\eta\psi(t,x,z)}dt\right)\hat{f}(z,\eta)dz$$

$$Q_4\hat{f}(x,\eta)=-(e^{\eta A(2\pi)}-1)^{-1}(e^{\eta B(2\pi)}-1)^{-1}\int_0^{2\pi}\left(\int_0^{2\pi} e^{\eta\psi(t,x,z)}dt\right)\hat{f}(z,\eta)dz$$

$$\psi(t,x,z)=A(x)-A(t)+B(t)-B(z).$$

Taking into account that $ReA(x)$ and $ReB(x)$ are nondecreasing we deduce 1) for Q_1: $Re\psi(t,x,z)\leq ReB(x)-ReB(z)\leq 0$ if $0\leq x\leq t\leq z\leq 2\pi$, 2) for Q_2: $Re\psi(t,x,z)\leq ReA(x)\leq ReA(2\pi)$ if $0\leq t\leq z\leq 2\pi$, $0\leq x\leq 2\pi$, 3) for Q_3: $Re\psi(t,x,z)\leq ReB(2\pi)-ReB(x)\leq ReB(2\pi)$ if $0\leq x\leq t\leq 2\pi$, $0\leq z\leq 2\pi$, 4) for Q_4: $Re\psi(t,x,z)\leq ReA(2\pi)+ReB(2\pi)$ if $0\leq t\leq 2\pi$, $0\leq x\leq 2\pi$, $0\leq z\leq 2\pi$. Clearly 1)–4) and (8) imply for $\eta\in N$

$$|Q\hat{f}(\cdot,\eta)|\leq q|\hat{f}(\cdot,\eta)|, \tag{9}$$

where $|\hat{f}(\cdot,\eta)|$ stands for the supremum norm of $\hat{f}(z,\eta)$. This proves the assertion (iii). The assertion (ii) follows from the fact that $Ker(P^*)$ consists of a function indentically equal to 1. The last part of the theorem is a consequence of the fact that under the assumptions on the constants c we can solve (7) if $|\eta|\geq 1$ while for $\eta=0$ c is not in the spectrum of D_x^2 on $\boldsymbol{T}^1$.

Proof of Theorem 3. Let us recall that if $\tau/2\pi$ is an irrational non Liouville number then there exist $C>0$, $k\in N$ such that $|e^{i\tau\eta}-1|=2|\sin(\tau\eta/2)|\geq C|\eta|^{-k}$, $\eta\in\boldsymbol{Z}$. Using these inequalities and the direct arguments in the previous theorem one establishes the estimate $|Q\hat{f}(\cdot,\eta)|\leq C|\eta|^k|\hat{f}(\cdot,\eta)|$, $|\eta|\geq 1$. This proves that $\hat{u}$ is smooth if $\hat{f}$ is smooth.

In order to prove the necessity, we construct a $\hat{f}(x,\eta)$ which is rapidly decreasing in η such that the equation has a non smooth solution. For this purpose, let us assume that $Rea(x)\not\equiv 0$ and $Reb(x)\equiv 0$ and that τ_b is a Liouville number. By replacing η by $-\eta$ if necessary, we may assume that $Rea(x)\geq 0$. We recall that we can prove (8) for large η. By definition, there exist $\{\eta_n\}$ $(\eta_n\to\infty)$ such that $e^{i\eta\tau_b}-1$ is rapidly decreasing for $\eta=\eta_n$, $n\to\infty$. Without loss of generality, we may assume that $\eta_n>0$. Let $g(\eta)$ be a function of η which is identically equal to zero when $\eta\neq\eta_n$, $n=1,2,\cdots$, and, for $\eta=\eta_n$, set $g(\eta)=e^{B(2\pi)\eta}-1$. We note that $g(\eta)$ is rapidly decreasing when $\eta\to\infty$. We set $\hat{f}(z,\eta)=g(\eta)e^{B(z)\eta}$. It follows from (8) that only the terms Q_2 and Q_4 appear in the expression of $\hat{u}(2\pi,\eta)$. By definition $E_2\hat{f}$ is rapidly decreasing in η. In order to study the term $E_4\hat{f}$ we have to estimate the integral of the form $\int_0^{2\pi}e^{-A(t)\eta+B(t)\eta}dt$. Without loss of generality, we may assume that $Rea(0)>0$. We divide the integral from 0 to m and from m to 2π for $m>0$ small. Then, in the integral from m to 2π we have that $ReA(t)\eta\geq c\eta+O(1)$, for $c>0$. Hence, the integral is exponentially decreasing. In order to estimate the first integral, we use Watson's lemma namely

$$\int_0^m s^{\mu-1}g(s)e^{-\eta s^r}ds \sim \sum_{j=0}^{\infty}\frac{g^{(j)}(0)}{j!r}\Gamma\left(\frac{j+\mu}{r}\right)\eta^{-(j+\mu)/r}, \quad \eta\to\infty,$$

where $\mu>0$, $r\geq 1$, $g(s)\in C_0^\infty[0,m)$ and Γ stands for the Euler gamma function. We easily see that $E_4\hat{f}=O(|\eta|^{-1})$. Hence it follows that $\hat{u}(2\pi,\eta)=O(|\eta|^{-1})$. This proves the assertion. The other cases will be proved by more detailed arguments.

Theorem 2 is proved by use of the following lemma and the condition $\delta>0$.

Lemma. *Let Rea do not change its sign and Rea$\not\equiv 0$. Then for every $f\in C^\infty(\boldsymbol{T}^2)$ verifying the integral othogonal condition, there exists an unique $u(x,y)$ verifying the same condition, which is a solution to the equation $L_a u=(D_x+ia(x)D_y)u=f$. Moreover, for any $s\in\boldsymbol{R}$, there exists $C>0$ such that $\|u\|_{s+1/(k_a+1)}\leq C\|f\|_s$. Here $\|\cdot\|_s$ stands for the $H^s(\boldsymbol{T}^2)$ norm.*

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