13. Levi Conditions for Hyperbolic Operators with a Stratified Multiple Variety

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1. Introduction and result. Let $P(x, D_x)$ be a differential operator of order m, i.e. $P(x, D_x)=P_m(x, D_x)+P_{m-1}(x, D_x)+\cdots$, where $P_j(x, D_x)$, $j=0, \cdots, m$, denotes the homogeneous part of order j of P (here $D_x=(1/i)\partial/\partial x$). We assume that P has C^∞ (smooth) coefficients in the open subset $\Omega\subset \boldsymbol{R}^{n+1}$, and that $0\in\Omega$. Consider the principal symbol of P, $p_m(x, \xi)$, which we shall assume to be a homogeneous polynomial of degree m with real valued smooth coefficients; we say that P is hyperbolic with respect to the direction ξ_0 if the equation $p_m(x, \xi)=0$, where $x=(x_0, x_1, \cdots, x_n)$, $\xi=(\xi_0, \xi_1, \cdots, \xi_n)$, has only real roots in ξ_0. It has long been well known that if P is strictly hyperbolic, i.e. if all the above mentioned roots of $p_m(x, \xi)=0$ are distinct, then the Cauchy problem

$$P(x, D_x)=f, \qquad \partial_0^j u|_{x_0=0}=g_j, \qquad j=0, \cdots, m-1,$$

is well posed. Well posedness, roughly speaking, means that there exists a unique distribution solution for any choice of the distributions f and g_j's. On the other hand, if the roots of $p_m(x, \xi)$ are not distinct, it is well known that in general we have well posedness only if we assume some conditions on the lower order terms, see e.g. [7] and [9] in the case of double roots, [10] and [11] in the case of roots of higher multiplicity.

When roots of higher multiplicity occur an important object is the localised principal symbol: If $d^j p_m(\rho)=0$, $j=0, \cdots, r-1$, and $d^r p_m(\rho)\neq 0$, define $p_{m,\rho}(\delta z)=\lim_{t\to 0} t^{-r} p_m(\rho+t\delta z)$, where $\delta z\in T_\rho(T^*\Omega)$, the tangent space at ρ of $T^*\Omega\simeq\Omega\times\boldsymbol{R}_\xi^{n+1}$.

In this note we present a result on necessary conditions for the well posedness of the Cauchy problem for P. Here is a list of the assumptions we make:

(H_1) The principal symbol $p_m(x, \xi)$ is real and hyperbolic with respect to ξ_0.

(H_2) The characteristic roots of $\xi_0\mapsto p_m(x, \xi_0, \xi')$ have multiplicity of order at most 3 and $\mathrm{Char}\, P=\{(x, \xi)\,|\, p_m(x, \xi)=0\}=\Sigma_1\cup\Sigma_2\cup\Sigma_3$, where

$$\Sigma_1=\{(x, \xi)\in T^*\Omega\,|\, p_m(x, \xi)=0,\ dp_m(x, \xi)\neq 0\},$$
$$\Sigma_2=\{(x, \xi)\in T^*\Omega\,|\, p_m(x, \xi)=0,\ dp_m(x, \xi)=0,\ d^2p_m(x, \xi)\neq 0\},$$
$$\Sigma_3=\{(x, \xi)\in T^*\Omega\,|\, p_m(x, \xi)=0,\ dp_m(x, \xi)=0,\ d^2p_m(x, \xi)=0\}.$$

Here and in the sequel $x'=(x_1, \cdots, x_n)$ and analogously for ξ'.

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(H_3) Let $\rho \in \Sigma_3$; then $p_{m,\rho}$, the localization of $p_m(x, \xi)$ at ρ defined, above, is a third order polynomial hyperbolic with respect to $(0, e_0)=(0, \cdots, 0, 1, 0, \cdots, 0)$ satisfying the following conditions:

(i) $p_{m,\rho}(\delta z)=L(\delta z)Q_2(\delta z)$, where $L(\delta z)=\delta\xi_0-l_1(\delta x, \delta\xi')$, l_1 being a real linear form in the variables $(\delta x, \delta\xi')$.

(ii) $Q_2(\delta z)$ is a real hyperbolic quadratic form such that

(a) $\ker F_{Q_2}^2 \cap \operatorname{Im} F_{Q_2}^2=\{0\}$, where $F_{Q_2}(\rho)=(d_{(x,\xi)}H_{Q_2})(\rho)$ and $H_{Q_2}(x, \xi)=(d_\xi Q_2, -d_x Q_2)(x, \xi)$ is the Hamilton vector field of Q_2.

(b) $dx \wedge d\xi|_{\operatorname{Im} F_{Q_2}}$ has positive rank.

(c) $\operatorname{sp}(F_{Q_2}) \subset i\boldsymbol{R}$ (this condition can be rephrased saying that Q_2 is non effectively hyperbolic).

(H_4) For every $\rho \in \Sigma_3$ define the lineality of p_m, $\Lambda_\rho(p_m)=\{\delta z \mid \delta z \in \ker F_{Q_2}(\rho),\ L(\delta z)=0\}$. Then $H_L(\rho) \in \Lambda_\rho(p_m)$.

To state the theorem we need some notation; if P is a hyperbolic polynomial and $\rho \in \operatorname{Char} P$, denote by $\Gamma_{p_m,\rho}=$the connected component of $\{\delta z \in T_\rho(T^*\Omega) \mid p_{m,\rho}(\delta z) \neq 0\}$ containing $(0, e_0)$, and by $\Gamma^\sigma_{p_m,\rho}=\{\delta z=(\delta x, \delta\xi) \in T_\rho(T^*\Omega) \mid \langle\delta\xi, \delta y\rangle-\langle\delta x, \delta\eta\rangle=d\xi \wedge dx(\delta x, \delta\xi; \delta y, \delta\eta) \geq 0,\ \forall(\delta y, \delta\eta) \in \Gamma_{p_m,\rho}\}$ its symplectic polar. Furthermore we shall denote by p^s the subprincipal symbol of P (see e.g. [8]) defined as $p^s(x, \xi)=p_{m-1}(x, \xi)+(i/2)\sum_{j=0}^n \partial^2_{x_j\xi_j}p_m(x, \xi)$ and by $\operatorname{Tr}^+F_{Q_2}=\sum \mu_j$, where $i\mu_j$ are the eigenvalues of F_{Q_2} on the positive imaginary axis, repeated according to their multiplicities.

We can now state our result:

Theorem 1. *Let $\Omega_t=\{x \in \Omega \mid x_0 \leq t\}$. Assume that the Cauchy problem for P is well posed in Ω_t, t small, and let $\rho \in \Sigma_3$. Assuming* (H_1)–(H_4), *the following conditions are necessary*:

$(\mathrm{L1})_l$ $$p^s(\rho)=0.$$

$(\mathrm{L2})_l$ $$\operatorname{Im} H_{p^s}(\rho)=0,$$
$$\operatorname{Tr}^+F_{Q_2}H_L \pm \operatorname{Re} H_{p^s}(\rho) \in \Gamma^\sigma_{p_m,\rho}.$$

Assumption $(H_4)_\rho$ is more general than the assumptions made in [3]; in the present case the localised polynomial need not be "strictly" hyperbolic with respect to $(0, e_0)$; in fact the relevant case we are interested in is the case when $H_L(\rho) \in \Lambda_\rho(p_m)\backslash\Gamma^\sigma_{Q_2}$, which is not covered in [3]. We would also like to point out that both assumptions (H_1)–(H_4) and the conditions on the lower order terms in (L1), (L2) are invariant under canonical transformations.

2. Examples. Here are two examples of operators satisfying hypotheses (H_1)–(H_4).

(a) $$P(x, D_x)=(D_0-lx_3D_n)(-D_0^2+x_1^2D_n^2+x_2^2D_n^2+D_2^2)$$
$$+a_0D_0D_n+a_2D_2D_n+(b_1x_1+b_2x_2+b_3x_3)D_n^2, \qquad |l|>1.$$

Here $p^s(x, \xi)=a_0\xi_0\xi_n+a_2\xi_2\xi_n+(b_1x_1+b_2x_2+b_3x_3)\xi_n^2$. Let $\rho=(\bar{x}_0, 0, 0, 0, \bar{x}_4, \cdots, \bar{x}_n; 0, \bar{\xi}_1, 0, \bar{\xi}_3, \cdots, \bar{\xi}_{n-1}, 1)$, so that (L1) is satisfied; condition (L2) with the $+$ sign means that $\operatorname{Im} a_j=0$, $j=0, 2$, $\operatorname{Im} b_j=0$, $j=1, 2, 3$ and

$$-\frac{b_3}{l}+1 \geq 0, \qquad a_0+\frac{b_3}{l} \geq \sqrt{a_2^2+b_1^2+b_2^2}.$$

(b) $$P(x, D_x)=(D_0-lD_1)(-D_0^2+D_1^2+x_2^2D_n^2+D_2^2)+(a_0D_0+a_1D_1+a_2D_2)D_n+b_2x_2D_n^2, \qquad |l|>1.$$

In this case $p^s(x, \xi)=(a_0\xi_0+a_1\xi_1+a_2\xi_2+b_2x_2\xi_n)\xi_n$. Let $\rho=(\bar{x}_0, \bar{x}_1, 0, \bar{x}_3, \cdots, \bar{x}_n; 0, 0, 0, \bar{\xi}_3, \cdots, \bar{\xi}_{n-1}, 1)$, so that (L1) is satisfied, condition (L2) with the $+$ sign means that $\operatorname{Im} a_j=0$, $j=0, 1, 2$, $\operatorname{Im} b_2=0$ and either

$$a_0+1\geq\sqrt{(a_1-l)^2+a_2^2+b_2^2}$$

or

$$a_0+1<\sqrt{(a_1-l)^2+a_2^2+b_2^2}, \qquad l^2-1\geq a_0+a_1l,$$
$$(a_0l+a_1)^2\geq(l^2-1)(a_2^2+b_2^2), \qquad l(a_0l+a_1)\geq 0.$$

3. **Proof of the theorem.** The proof of the theorem is done in two parts according to the mutual positions of H_L and $\Gamma^\sigma_{Q_2}$; actually we distinguish two cases: i) $H_L\notin \operatorname{Im} F_{Q_2}$ and ii) $H_L\in \operatorname{Im} F_{Q_2}\backslash\Gamma^\sigma_{Q_2}$. A model of these two cases is given by Examples (a) and (b) respectively.

Case i) has the following geometrical consequence:

Proposition 2. *If* $H_L\notin \operatorname{Im} F_{Q_2}$ *then*

$$\partial\Gamma_{Q_2}=\Lambda(Q_2)+(\partial\Gamma_{Q_2}\cap\Lambda(L)).$$

The above equality expresses the fact that the double characteristic set $\tilde{\Sigma}_2$ of the localised operator is "large enough" to generate the hyperbolicity cone Γ_{Q_2}; this is made more precise by the following

Proposition 3. *For every* $z\in\Gamma_{Q_2}$ *there exists*

$$v_1\in\{(\delta x, \delta\xi)\in T_\rho(T^*\Omega)\,|\,Q_{2,\rho}(\delta x, \delta\xi)=0,\ dQ_{2,\rho}(\delta x, \delta\xi)=0,\ L(\delta x, \delta\xi)\neq 0\},$$

and

$$v_2\in\{(\delta x, \delta\xi)\in T_\rho(T^*\Omega)\,|\,Q_{2,\rho}(\delta x, \delta\xi)=0,\ dQ_{2,\rho}(\delta x, \delta\xi)\neq 0,\ L(\delta x, \delta\xi)=0,\ \delta\xi_0>0\},$$

such that

$$z=\lambda_1v_1+\lambda_2v_2,$$

where $\lambda_i\geq 0$, $i=1, 2$.

The proof in case i) is accomplished by constructing an asymptotic solution whose phase presents a double scaling: one of these scaling allows us to microlocalise near the triple point ρ under consideration, whereas the other can be thought of as a kind of second microlocalisation along the double manifold of $p_{m,\rho}$.

Let us now turn to case ii). Here, in contrast to the preceding situation, the double characteristic set of $p_{m,\rho}$ is too small to generate the hyperbolicity cone Γ_{Q_2}. Therefore Propositions 2 and 3 are no longer true and the proof of the theorem can rely only in part on $\tilde{\Sigma}_2$; further analysis of the triple characteristic set, refining the technique developed in [3], must be used in order to get the full Levi conditions (L1) and (L2).

The details will appear elsewhere (see [5]).

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