10. *The Aitken-Steffensen Formula for Systems of Nonlinear Equations. V*

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1. Introduction. Let $x=(x_1, x_2, \cdots, x_n)$ be a vector in R^n and D a region contained in R^n. Let $f_i(x)$ $(1\leqq i\leqq n)$ be real-valued nonlinear functions defined on D and $f(x)=(f_1(x), f_2(x), \cdots, f_n(x))$ an n-dimensional vector-valued function. Then we shall consider a system of nonlinear equations

$$x=f(x), \tag{1.1}$$

whose solution is $\bar{x}$.

As mentioned in [2]–[4], Henrici [1, p. 116] has considered a formula, which is called the Aitken-Steffensen formula. Now, we have studied the above Aitken-Steffensen formula for systems of nonlinear equations in [2]–[4], and shown [2, Theorem 2], [3, Theorem 2] and [4, Theorem 1].

The purpose of this paper is to show Theorem 4 by combining [2, Theorem 2] with [2, Theorem 1], and Theorem 5 by using only the relation in [4, Theorem 1].

2. Statement of results. For any $x\in R^n$ and an $n\times n$ matrix $A=(a_{ij})$, we shall use the norms $\|x\|$ and $\|A\|$ defined by

$$\|x\|=\max_{1\leqq i\leqq n}|x_i| \quad\text{and}\quad \|A\|=\max_{1\leqq i\leqq n}\sum_{j=1}^{n}|a_{ij}|,$$

respectively. Let $U(\bar{x})=\{x\,;\,\|x-\bar{x}\|<\delta\}\subset D$ be a neighbourhood.

Given $x^{(0)}\in R^n$, define $x^{(i)}\in R^n$ $(i=1,2,\cdots)$ by

$$x^{(i+1)}=f(x^{(i)}) \quad (i=0,1,2,\cdots). \tag{2.1}$$

Put

$$d^{(i)}=x^{(i)}-\bar{x} \qquad \text{for } i=0,1,2,\cdots, \tag{2.2}$$

and then define an $n\times n$ matrix D_k by

$$D_k=(d^{(k)}, d^{(k+1)}, \cdots, d^{(k+n-1)}).$$

Throughout this paper, we shall assume the same conditions (A.1)–(A.5) as in [2].

(A.1) $f_i(x)$ $(1\leqq i\leqq n)$ are two times continuously differentiable on D.

(A.2) There exists a point $\bar{x}\in D$ satisfying (1.1).

(A.3) $\|J(\bar{x})\|<1$, where $J(x)=(\partial f_i(x)/\partial x_j)$ $(1\leqq i,\ j\leqq n)$.

(A.4) The vectors $d^{(k)}, d^{(k+1)}, \cdots, d^{(k+n-1)}$, $k=0, 1, 2, \cdots$, are linearly independent.

(A.5) $\inf\{|\det D_k|/\|d^{(k)}\|^n\}>0$.

Then, we shall consider the Aitken-Steffensen formula

$$y^{(k)}=x^{(k)}-\varDelta X^{(k)}(\varDelta^2X^{(k)})^{-1}\varDelta x^{(k)} \quad (k=0,1,2,\cdots), \tag{2.3}$$

where an n-dimensional vector $\Delta x^{(k)}$, and $n\times n$ matrices $\Delta X^{(k)}$ and $\Delta^2 X^{(k)}$ are given by

(2.4) $$\Delta x^{(k)}=x^{(k+1)}-x^{(k)},$$

(2.5) $$\Delta X^{(k)}=(x^{(k+1)}-x^{(k)},\cdots,x^{(k+n)}-x^{(k+n-1)})$$

and

(2.6) $$\Delta^2 X^{(k)}=\Delta X^{(k+1)}-\Delta X^{(k)}.$$

Now, we have shown the following Theorems 1 and 2 in [2], and Theorem 3 in [4].

Theorem 1 ([2, Theorem 1]). *Under conditions* (A.1)–(A.3), *we have*

(2.7) $$\|x^{(k+1)}-\bar{x}\|\leqq M_1\|x^{(k)}-\bar{x}\| \quad (k=0,1,2,\cdots)$$

for any $x^{(0)}\in U(\bar{x})$ *and a constant* M_1 *with* $\|J(\bar{x})\|<M_1<1$.

Theorem 2 ([2, Theorem 2]). *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *there exists a constant* M_2 *such that the property*

(2.8) $$\|y^{(k)}-\bar{x}\|\leqq M_2\|x^{(k)}-\bar{x}\|^2$$

holds for sufficiently large k.

Theorem 3 ([4, Theorem 1]). *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *a new relation of the form*

(2.9) $$\|y^{(k+1)}-\bar{x}\|\leqq M\|y^{(k)}-\bar{x}\|+\varepsilon_k, \qquad \varepsilon_k\to 0\ (k\to\infty)$$

holds with a constant M *satisfying* $\|J(\bar{x})\|<M<1$, *where* ε_k *can be considered as "convergent term".*

In this paper, we show the following results.

Theorem 4. *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *Theorem* 2, *together with Theorem* 1, *implies*

$$y^{(k)}\to\bar{x} \quad as \quad k\to\infty.$$

Theorem 5. *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in U(\bar{x})$, *Theorem* 3 *implies*

$$y^{(k)}\to\bar{x} \quad as \quad k\to\infty.$$

As seen above, the result of Theorem 4 is the same as that of Theorem 5, but we show Theorem 4 by combining (2.8) in Theorem 2 with (2.7) in Theorem 1, and Theorem 5 by using only the relation (2.9) in Theorem 3.

3. Preliminaries. By (2.2) and (2.4), we have

(3.1) $$\Delta x^{(k)}=d^{(k+1)}-d^{(k)},$$

and, by (2.1), (2.2) and (A.2),

(3.2) $$d^{(k+1)}=J(\bar{x})d^{(k)}+\xi(x^{(k)}),$$

$\xi(x^{(k)})$ being an n-dimensional vector, and by (A.1),

(3.3) $$\|\xi(x^{(k)})\|\leqq L_1\|d^{(k)}\|^2 \qquad \text{for } x^{(k)}\in U(\bar{x}),$$

a constant L_1 being suitably chosen. Then, from (3.1) by using (3.2), (3.3) and (A.3), we see that the inequality

(3.4) $$\|\Delta x^{(k)}\|\leqq L_2\|d^{(k)}\|$$

holds with a constant L_2 chosen suitably.

For the proof of Theorem 5, we need the following lemma given in [2].

Lemma 1 ([2, Lemma 4]). *Under conditions* (A.1)–(A.5), *for* $x^{(k)}\in$

$U(\bar{x})$, the $n\times n$ matrix $\Delta^2 X^{(k)}$ given by (2.6) *is invertible, and there exists a constant L_3 such that the inequality*

(3.5) $$\|(\Delta^2 X^{(k)})^{-1}\| \leqq L_3 \|d^{(k)}\|^{-1}$$

holds for sufficiently large k.

By (3.2), we have

$$d^{(k+i)} - d^{(k+i-1)} = (J(\bar{x}) - I) d^{(k+i-1)} + \xi(x^{(k+i-1)}),$$

so that

(3.6) $$\Delta X^{(k)} = (J(\bar{x}) - I) D_k + (\xi(x^{(k)}), \cdots, \xi(x^{(k+n-1)}))$$

follows from (2.5).

We note that [2, Theorem 1] leads to

(3.7) $$\|D_k\| \leqq \sum_{i=0}^{n-1} \|d^{(k+i)}\| \leqq n \|d^{(k)}\|.$$

Since

$$\|(\xi(x^{(k)}), \cdots, \xi(x^{(k+n-1)}))\| \leqq \sum_{i=1}^{n} \|\xi(x^{(k+i-1)})\|,$$

we have, by (3.3) and [2, Theorem 1],

(3.8) $$\|(\xi(x^{(k)}), \cdots, \xi(x^{(k+n-1)}))\| \leqq L_4 \|d^{(k)}\|^2 \qquad \text{for } x^{(k)} \in U(\bar{x}),$$

a constant L_4 being suitably chosen. Then using $\|I\| = 1$, there exists a constant L_5 such that the inequality

(3.9) $$\|\Delta X^{(k)}\| \leqq L_5 \|d^{(k)}\|$$

holds for $x^{(k)} \in U(\bar{x})$, from (3.6), by (A.3), (3.7) and (3.8).

4. Proofs of Theorems 4 and 5. We shall prove Theorems 4 and 5.

Proof of Theorem 4. By repeating the process (2.7) in Theorem 1, we have

$$\|x^{(k)} - \bar{x}\| \leqq M_1^k \|x^{(0)} - \bar{x}\|$$

for any $x^{(0)} \in U(\bar{x})$, and so combined with (2.8) in Theorem 2, we obtain

$$\|y^{(k)} - \bar{x}\| \leqq M_2 M_1^{2k} \|x^{(0)} - \bar{x}\|^2 \to 0 \quad \text{as} \quad k \to \infty,$$

since $\|J(\bar{x})\| < M_1 < 1$. This proves the theorem.

Proof of Theorem 5. We recall that (3.5) in Lemma 1 holds, provided that k is sufficiently large. Now, (2.3) gives

(4.1) $$\|y^{(k)} - \bar{x}\| \leqq \|d^{(k)}\| + \|\Delta X^{(k)}\| \|(\Delta^2 X^{(k)})^{-1}\| \|\Delta x^{(k)}\|.$$

Then by (4.1) with (3.4), (3.5) and (3.9), for $x^{(k)} \in U(\bar{x})$, there exists a constant $K_1 > 0$ such that

(4.2) $$\|y^{(k)} - \bar{x}\| \leqq (1 + L_2 L_3 L_5) \delta$$

holds for any integer $k > K_1$.

Since $\varepsilon_k \to 0$ $(k \to \infty)$ in (2.9), it follows that for an arbitrary but fixed $\varepsilon > 0$, there exists a constant $K_2 > 0$ such that

(4.3) $$0 \leqq \varepsilon_k < \varepsilon$$

for any integer $k > K_2$. So putting $N = \max(K_1, K_2)$, we have (4.2) and (4.3) for $k > N$.

By repeating the process (2.9) in Theorem 3, we obtain

(4.4) $$\|y^{(k+1)} - \bar{x}\| \leqq M^{k-N} \|y^{(N+1)} - \bar{x}\| + \sum_{i=0}^{k-N-1} M^i \varepsilon_{k-i},$$

and, from (4.4), by (4.2) and (4.3),

$$\|y^{(k+1)}-\bar{x}\| \leqq M^{k-N}(1+L_2L_3L_5)\delta+\frac{\varepsilon}{1-M} \tag{4.5}$$

for $k>N$.

As M was chosen so as to satisfy $\|J(\bar{x})\|<M<1$, we see that there exists a constant $K>N$ such that

$$M^{k-N}<\varepsilon \tag{4.6}$$

for $k>K$. Therefore, for $x^{(k)} \in U(\bar{x})$,

$$\|y^{(k+1)}-\bar{x}\| \leqq \left[(1+L_2L_3L_5)\delta+\frac{1}{1-M}\right]\varepsilon$$

follows from (4.5) by using (4.6), provided $k>K$. This proves our Theorem 5. In this way, we have proved Theorems 4 and 5, as desired.

Remark 1. We note that, for $x^{(k)} \in U(\bar{x})$,

$$\|\Delta x^{(k)}\| \leqq (M_1+1)\|d^{(k)}\|$$

holds from (3.1), by Theorem 1. So we can take M_1+1 as L_2 in (3.4).

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