84. Remarks to our Former Paper, "Uniform Distribution of Some Special Sequences"

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Abstract: In [2], Y. H. Too pointed out that our proof of Theorem 2 of our former paper [1] contained an error. In this paper, we shall first restate the main results of [1], [2] as Theorems A, B, C, then give a revised proof of Theorem B (= Theorem 2 [1]), prove a Proposition which, combined with Theorem A (= Theorem 1 [1] which was correctly proved), yields Theorem C and finally remark that Theorem B can also be easily deduced from Theorem A.

Let p_n be the n-th prime number.

Theorem A (Theorem 1 [1]). *Let $f(x)$ be a continuously differentiable function with $f(x) \to \infty$ $(x \to \infty)$. If $f'(x)\log x$ is monotone, $n|f'(n)| \to \infty$ as $n \to \infty$, and*

$$f(n)/(\log n)^l \to 0 \quad (n \to \infty) \text{ for some } l > 1,$$

then $(\alpha f(p_n))$ is uniformly distributed mod 1, *where $\alpha(\neq 0)$ is any real constant.*

Theorem B (Theorem 2 [1]). *Let $f(x)$ be a continuously differentiable function with $f'(t) > 0$ and $f''(t) > 0$. If $t^2 f''(t) \to \infty$ as $t \to \infty$ and*

$$f(n)/(\log n)^l \to 0 \quad (n \to \infty) \quad \text{for some } l > 1,$$

then $(\alpha f(p_n))$ is uniformly distributed mod 1, *where $\alpha(\neq 0)$ is any real constant.*

Theorem C (Theorem 3 [2]). *Let f be a twice differentiable function with $f \to \infty$, $f' > 0$ and $f'' < 0$. If $x^2(-f''(x)) \to \infty$, $(\log x)^2(-f''(x))$ is decreasing as $x \to \infty$ and $f(n)/(\log n)^l \to 0 (n \to \infty)$ for some $l > 1$, then $(\alpha f(p_n))_1^\infty$ is uniformly distributed* mod 1, *where $\alpha(\neq 0)$ is any real constant.*

Revised proof of Theorem B. The proof becomes correct if we change the estimation of I_2 in [1 : p.84 line 6 ↑ through p.85 line 3] as follows:

We choose any sequence $c_N \to \infty$ as $N \to \infty$, and put

$$I_2 = \int_2^{p_N} \frac{e^{2\pi i h f(t)}}{\log t}\, dt = \left(\int_2^{c_N} + \int_{c_N}^{p_N}\right) \frac{e^{2\pi i h f(t)}}{\log t}\, dt = A + B, \text{ say.}$$

Then clearly

$$|A| = \left| \int_2^{c_N} \frac{e^{2\pi i h f(t)}}{\log t}\, dt \right| \leq \int_2^{c_N} \frac{dt}{\log t} \ll \frac{c_N}{\log c_N}.$$

Now applying [3 : Lemma 10.2], we get

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$$|B| = \left| \int_{c_N}^{p_N} \frac{e^{2\pi i h f(t)}}{\log t} dt \right| \ll \max_{c_N \le t \le p_N} \frac{1}{\log t \sqrt{h |f''(t)|}}$$

$$= \frac{1}{\sqrt{h}} \max_{c_N \le t \le p_N} \frac{t}{\log t} \frac{1}{\sqrt{t^2 |f''(t)|}}$$

$$\ll \frac{1}{\sqrt{h}} \frac{p_N}{\log p_N} \max_{c_N \le t \le p_N} \frac{1}{\sqrt{t^2 |f''(t)|}} = \frac{N}{\sqrt{h}} o(1),$$

since $p_N \sim N \log N$ and $c_N \to \infty$ as $N \to \infty$.

Thus by the Erdös-Turán inequality for the discrepancy D_N of $f(p_N)$ we have

$$D_N \ll \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \left(\frac{p_N}{(\log p_N)^k} + \frac{c_N}{\log c_N} + \frac{N}{\sqrt{h}} o(1) + \frac{h p_N}{(\log p_N)^k} f(p_N) \right) \right|$$

$$\ll \frac{1}{m} + \frac{p_N}{N (\log p_N)^k} \log m + \frac{c_N}{N \log c_N} \log m + \sum_{h=1}^{m} \frac{1}{h\sqrt{h}} o(1)$$

$$+ \frac{p_N}{N (\log p_N)^k} f(p_N) m.$$

Choosing $m = \log N$ and $c_N = \sqrt{N}$, we have $D_N = o(1)$, which proves Theorem B.

Proposition (see, Theorem 3 [2]). *Let $f(x)$ be a twice differentiable function with $f' > 0$ and $f'' < 0$. If $x^2(-f''(x)) \to \infty$, then $xf'(x) \to \infty$. If $x(\log x)^2(-f''(x))$ is decreasing, then $(\log x)f'(x)$ is monotone. Moreover $(\log x)f'(x)$ is decreasing or increasing according as $f'(x)$* tends to zero or to a positive constant.

Proof. Since $f'' < 0$ and $f' > 0$, in case $f'(x) \to m > 0$, we have $xf'(x) \to \infty$. Otherwise by L'Hospital's rule, we have $xf'(x) \to \infty$.

Next, we set $M(x) = x(\log x)^2(-f''(x)) > 0$. Since $M(x)$ is decreasing and bounded from below, we have $\lim_{x\to\infty} M(x) = M$. For any positive ε, there exists c such that for any $x > c$, $M \le M(x) < M + \varepsilon$. Now

$$-f''(x) = \frac{M(x)}{x(\log x)^2}$$

$$(*) \qquad -f'(x) = \int_c^x \frac{M(t)}{t(\log t)^2} dt - f'(c).$$

Therefore

$$T(x) := (\log x)f'(x) = -(\log x) \int_c^x \frac{M(t)}{t(\log t)^2} dt + (\log x)f'(c).$$

Thus

$$T'(x) = -\frac{1}{x} \int_c^x \frac{M(t)}{t(\log t)^2} dt - \frac{M(x)}{x(\log x)} + \frac{f'(c)}{x} \ge -\frac{1}{x} \int_c^x \frac{M+\varepsilon}{t(\log t)^2} dt$$

$$- \frac{M+\varepsilon}{x(\log x)} + \frac{f'(c)}{x}$$

$$= -\frac{M+\varepsilon}{x}\left(\frac{1}{\log c}-\frac{1}{\log x}\right)-\frac{M+\varepsilon}{x(\log x)}+\frac{f'(c)}{x}=-\frac{1}{x}\frac{M+\varepsilon}{\log c}+\frac{f'(c)}{x}$$

$$=\frac{1}{x}\left(f'(c)-\frac{M+\varepsilon}{\log c}\right)>0,$$

if $\lim_{x\to\infty} f'(x) = m > 0$ and c being sufficiently large.

If $m = 0$, then from $(*)$,

$$f'(c) = \int_c^\infty \frac{M(t)}{t(\log t)^2}\,dt,$$

and

$$T(x) = (\log x)\int_x^\infty \frac{M(t)}{t(\log t)^2}\,dt.$$

Thus

$$T'(x) = \frac{1}{x}\int_x^\infty \frac{M(t)}{t(\log t)^2}\,dt - \frac{M(x)}{x}\frac{1}{\log x}$$

$$=\frac{1}{x}\int_x^\infty \frac{M(t)}{t(\log t)^2}\,dt-\frac{M(x)}{x}\int_x^\infty \frac{dt}{t(\log t)^2}=\frac{1}{x}\int_x^\infty \frac{M(t)-M(x)}{t(\log t)^2}\,dt\le 0,$$

since $M(x)$ is monotonely decreasing. This completes the proof.

Remark 1. From this Proposition, we can obtain Theorem C from Theorem A.

Remark 2. Our Theorem 2 [1] can be also deduced from our Theorem 1 [1] as follows:

If $t^2 f''(t)\to\infty$, then we have $f(t)\to\infty$ as $t\to\infty$. Since $f''(t)>0$, $f'(t)$ is monotonely increasing. As $\log x$ is also monotonely increasing, $f'(x)\log x$ is monotonely increasing. Hence we have $n\,|f'(n)|\to\infty$ as $n\to\infty$, because $f'(x)$ is monotonely increasing and $f'(t)>0$.

References

[1] K. Goto and T. Kano: Uniform distribution of some special sequences. Proc. Japan Acad., **61A**, 83–86 (1985).

[2] Y. H. Too: On the uniform distribution modulo one of some log-like sequences. ibid., **68A**, 269–272 (1992).

[3] A. Zygmund: Trigonometric Series. vol.1, Cambridge, 1959.