80. *Primitive π-regular Semigroups*

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 14, 1992)

Abstract: In this note we investigate the structure of π-regular semigroups, the nonzero idempotents of which are primitive.

Various characterizations for primitive regular semigroups have been obtained by T. E. Hall [4], G. Lallement and M. Petrich [6], G. B. Preston [7], O. Steinfeld [8] and P. S. Venkatesan [9], [10](this appeared also in the book of A. H. Clifford and G. B. Preston [3]). J. Fountain [5] considered primitive abundant semigroups. In this paper we consider primitive π-regular semigroups and in this way we generalize the previous results for primitive regular semigroups.

Throughout this paper, $\boldsymbol{Z}^+$ will denote the set of all positive integers. If S is a semigroup with zero 0, we will write $S = S^0$ and $S^* = S - \{0\}$.

An element a of a semigroup $S = S^0$ is a *nilpotent* if there exists $n \in \boldsymbol{Z}^+$ such that $a^n = 0$. The set of all nilpotents of a semigoup S is denoted by $Nil(S)$. A semigroup S is a *nil-semigroup* if $S = Nil(S)$. An ideal I of a semigroup $S = S^0$ is a *nil-ideal* of S if I is a nil-semigroup. An ideal extension S of a semigroup K is a *nil-extension* of K if S/K is a nil-semigroup. By $R^*(S)$ we denote *Clifford's radical* of a semigroup $S = S^0$, i.e. the union of all nil-ideals of S (it is the greatest nil-ideal of S).

A semigroup S is *π-regular* (*completely π-regular*) if for every $a \in S$ there exist $n \in \boldsymbol{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$ ($a^n = a^n x a^n$ and $a^n x = x a^n$). A semigroup S is *π-inverse* if S is *π-regular* and every regular element of S has a unique inverse. If A is a nonempty subset of a semigroup S, then by $\mathrm{Reg}(A)$ $(E(A))$ we denote the set of all regular elements (idempotents) of A. If e is an idempotent of a semigroup S then we denote by G_e the maximal subgroup of S with e as its identity. A nonzero idempotent e of a semigroup $S = S^0$ is *primitive* if for every $f \in E(S^*)$, $f = ef = fe \Rightarrow f = e$, i.e. if e is minimal in $E(S^*)$, relative to the partial order on $E(S^*)$. A semigroup $S = S^0$ is *primitive* if all of its nonzero idempotents are primitive.

For undefined notion and notations we refer to [2] and [3].

Lemma 1. *Let $S = S^0$ be a semigroup. If eS (Se) is a 0-minimal right (left) ideal of S generated by a nonzero idempotent e, then e is primitive.*

Proof. For a proof see Lemma 6.38 [3].

The converse of the previous lemma is not true. For example, in the semigroup $S = \langle a, e, 0 \mid a^2 = 0,\ e^2 = e,\ ae = 0,\ ea = a,\ a0 = 0a = e0 =$

Supported by Grant 0401A of RFNS through Math. Inst. SANU.

$0e = 0^2 = 0\rangle$, e is a primitive idempotent. But $eS = S$, so eS is not a 0-minimal right ideal of S.

Now we introduce the following

Definition 1. A nonzero idempotent e of a semigroup $S = S^0$ which generates 0-minimal left (right) ideal is called *left* (*right*) *completely primitive*. An idempotent e is *completely primitive* if it is both left and right completely primitive.

A semigroup S is (*left, right*) *completely primitive* if all of its nonzero idempotents are (left, right) completely primitive.

For regular semigroups we have the following

Lemma 2 [3]. *Let $S = S^0$ be a regular semigroup and let $e \in E(S^*)$. Then e is primitive if and only if $eS\,(Se)$ is a 0-minimal left* (*right*) *ideal of S.*

Therefore, in regular semigroups the notions "primitive" and "completely primitive" coincide.

Lemma 3. *Let $S = S^0$ be a primitive π-regular semigroup. Then S is completely π-regular with maximal subgroups given by*

$$G_e = eSe - N,$$

where $e \in E(S^*)$ and $N = \mathrm{Nil}(S)$.

Proof. For a proof see Lemma 1 [1].

Theorem 1. *The following conditions on a semigroup $S = S^0$ are equivalent*:

(i) *S is a nil-extension of a primitive regular semigroup;*

(ii) *S is a completely primitive π-regular semigroup;*

(iii) *S is completely π-regular and SeS is a 0-minimal ideal of S for every $e \in E(S^*)$;*

(iv) *S is a primitive π-regular semigroup and $R^*(SE(S)S) = \{0\}$.*

Proof. (i) ⇒ (ii). Let S be a nil-extension of a primitive regular semigroup T. Assume $e \in E(S^*)$. Then

$$eS = e^2S \subseteq eTS \subseteq eT \subseteq eS,$$

whence $eS = eT$. By Lemma 2 we obtain that eT is a 0-minimal right ideal of T, and of S also. Therefore, S is right completely primitive. Similarly it can be proved that S is left completely primitive. It is clear that S is π-regular. Thus, (ii) holds.

(ii) ⇒ (i). Let S be a π-regular completely primitive semigroup. Let

$$R = \bigcup_{e \in E} eS, \quad L = \bigcup_{e \in E} Se, \quad E = E(S).$$

It is easy to verify that R is a right ideal and L is a left ideal of S. Since $eS \subseteq R$, $Se \subseteq L$, for every $e \in E(S^*)$, then by hypothesis we obtain that $eS = eR$ and $Se = Le$, whence

$$R = \bigcup_{e \in E} eR, \quad L = \bigcup_{e \in E} Le.$$

By Theorem 6.39 [3] it follows that R and L are primitive regular semigroups. Thus, $R, L \subseteq \mathrm{Reg}(S)$. Assume $a \in \mathrm{Reg}(S^*)$. Then $a = eaf$ for some $e, f \in E(S^*)$, whence $a \in eS \cap Sf \subseteq R \cap L$. Thus $\mathrm{Reg}(S) \subseteq R \cap L$. Therefore, $\mathrm{Reg}(S) = R = L$ is an ideal of S, and since for every $a \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in \mathrm{Reg}(S)$, we have that S is a nil-extension of a primitive regular semigroup.

(i) $\Rightarrow$ (iv). Let S be a nil-extension of a regular primitive semigroup T. It is clear that S is primitive and π-regular and that $T = SE(S)S$. Since T has not nonzero nil-ideals, we have $R^*(SE(S)S) = R^*(T) = \{0\}$.

(iv) $\Rightarrow$ (iii). Let S be a primitive π-regular semigroup and let $R^*(SE(S)S) = \{0\}$. Assume $e \in E(S^*)$. Let I be a nonzero ideal of S contained in SeS. Then I is an ideal of $SE(S)S$, so by the hypothesis we obtain that I is not a nil-ideal, so there exists $a \in I - \mathrm{Nil}(S)$. Moreover, there exists $n \in \boldsymbol{Z}^+$ and $x \in S$ such that $a^n = a^n x a^n$. Let $f = a^n x$. Then $f \in E(S^*)$ and by $a^n \in I$ it follows that $f \in I \subseteq SeS$, so $f = uev$ for some $u, v \in S$. Let $g = evfue$. Then $g^2 = g = ge = eg$ and $ugv = f$, so $g \neq 0$. By the primitivity of e we obtain that $g = e$, whence

$$e = evfue \in SfS \subseteq SIS \subseteq I.$$

Thus $SeS \subseteq I$, i.e. $SeS = I$. Therefore, SeS is a 0-minimal ideal of S.

By Lemma 3 it follows that S is completely π-regular.

(iii) $\Rightarrow$ (i). Let (iii) hold and let

$$T = SE(S)S = \bigcup_{e \in E} SeS, \quad E = E(S).$$

For $a \in \mathrm{Reg}(S^*)$ we have that $a = ea$ for some $e \in E(S^*)$, so $a = ea \in SeS \subseteq T$, Thus, $\mathrm{Reg}(S) \subseteq T$. Since S is completely π-regular, then for all $e \in E(S^*)$, SeS is also completely π-regular, so we obtain by Munn's theorem ([2], Theorem 2.55) that SeS is a completely 0-simple semigroup. Thus, $T \subseteq \mathrm{Reg}(S)$, i.e. $\mathrm{Reg}(S) = T$. Therefore, S is a nil-extension of a primitive regular semigroup $T = \mathrm{Reg}(S)$.

Lemma 4. *Let $S = S^0$ be a semigroup. Then*

$$R^*(S/R^*(S)) = \{0\}.$$

Proof. Let $S/R^*(S) = Q$, Let $\varphi : S \to Q$ be the natural homomorphism and let I be a nil-ideal of Q. Assume $J = \{x \in S \mid \varphi(x) \in I\}$. Then it is easy to verify that J is a nil-ideal of S, whence $J \subseteq R^*(S)$, so I is the zero ideal of Q.

We can now prove the structural theorem for primitive regular semigroups:

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a primitive π-regular semigroup;*

(ii) *S is an ideal extension of a nil-semigroup by a completely primitive π-regular semigoup;*

(iii) *S is a nil-extension of a semigroup which is an ideal extension of a nil-semigroup by a primitive regular semigroup.*

Proof. (i) $\Rightarrow$ (ii). Let S be a primitive π-regular semigroup. Then it is clear that $S/R^*(S)$ is a primitive π-regular semigroup, so by Lemma 4 and Theorem 1, we obtain that $S/R^*(S)$ is compeletely primitive. Thus, (ii) holds.

(ii) $\Rightarrow$ (i). Let S be an ideal extension of a nil-semigroup T by a completely primitive π-regular semigroup Q. Let us identify partial semigroups $S - T$ and Q^*. Assume $a \in S$. If $\langle a \rangle \subseteq S - T$, then $\langle a \rangle \subseteq Q^*$ in Q, so there exists $n \in \boldsymbol{Z}^+$ and $x \in Q^*$ such that $a^n = a^n x a^n$ in Q, whence $a^n =$

$a^n x a^n$ in S. If $\langle a \rangle \cap T \neq \phi$, then a is a nilpotent, so it is π-regular. It is clear that S is primitive. Therefore, S is a primitive π-regular semigroup.

(i) $\Rightarrow$ (iii). Let S be a primitive π-regular semigroup and let $K = SES$, where $E = E(S)$. Since $\mathrm{Reg}(S) \subseteq K$ and S is π-regular, then S is a nil-extension of K. Let $R = R^*(K)$, $Q = K/R$ and $E' = E(Q)$. Let $x \in Q$. Then $x = \varphi(a)$ for some $a \in K$ and φ is the natural homomorphism of K onto Q. Since

$$KEK \subseteq SES \subseteq SE^2 EE^2 S \subseteq (SES)E(SES) = KEK,$$

thus $K = KEK$. We have $a = uev$ for some $u, v \in K$, $e \in E$, whence

$$x = \varphi(a) = \varphi(u)\varphi(e)\varphi(v) \in QE'Q.$$

Hence $Q = QE'Q$. Since $R^*(Q) = R^*(QE'Q) = 0$ and Q is primitive π-regular, it follows from the proof of Theorem 1 that Q is a primitive regular semigroup.

(iii) $\Rightarrow$ (i). Let S be a nil-extension of a semigroup T and let T be an ideal extension of a nil-semigroup R by a primitive regular semigroup Q. Since we can identify partial semigroups $E(S) = E(T)$ and $E(Q)$, so S is primitive. It is clear that S is π-regular. Thus (i) holds.

Corollary 1. *A semigroup $S = S^0$ is a completely primitive π-inverse semigroup if and only if S is a nil-extension of a primitive inverse semigroup.*

Corollary 2. *The following conditions on a semigroup S are equivalent*:

(i) *S is a primitive π-inverse semigroup*;

(ii) *S is an ideal extension of a nil-semigroup by a completely primitive π-inverse semigroup*;

(iii) *S is a nil-extension of a semigroup which is an ideal extension of a nil-semigroup by a primitive inverse semigroup.*

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