9. *The Structure of Compactifications of C^3*

By Mikio FURUSHIMA
Department of Mathematics, College of Education, University of the Ryukyu

(Communicated by Heisuke HIRONAKA, M. J. A., Feb. 12, 1992)

Introduction. Let (X, Y) be a smooth projective compactification of C^3 with the second Betti number $b_2(X)=1$. Then Y is an irreducible ample divisor on X with $\operatorname{Pic} X \cong Z\mathcal{O}_X(Y)$ and the canonical divisor K_X can be written as $K_X \sim -rY$ $(r>0, r \in Z)$ (cf. [1]). Thus X is a Fano threefold of first kind (cf. [6]). The integer r is called the index of X.

Two smooth compactifications (X, Y) and (X', Y') are said to be isomorphic, denoted by $(X, Y) \cong (X', Y')$, if there is a biregular morphism $\alpha : X \to X'$ such that $\alpha(Y)=Y'$.

Then we have:

Theorem. (1) $r \geq 4 \Rightarrow (X, Y) \cong (P^3, P^2)$, *in fact*, $r=4$;

(2) $r=3 \Rightarrow (X, Y) \cong (Q^3, Q_0^2)$,

(3) $r=2 \Rightarrow (X, Y) \cong (V_5, H_5^0)$ *or* (V_5, H_5^∞),

(4) $r=1 \Rightarrow (X, Y) \cong (V_{22}, H_{22}^0)$ *or* (V_{22}, H_{22}^∞).

Remark 1. (1) (P^3, P^2), (Q^3, Q_0^2), (V_5, H_5^0), (V_5, H_5^∞) are determined uniquely up to isomorphism (cf. [5], [8]).

(2) (V_{22}, H_{22}^0), (V_{22}, H_{22}^∞) are not unique, in fact, they have a 4-dimensional family ([7]).

Notation. Q^3: a smooth quadric hypersurface in P^4

Q_0^2: a quadric cone in P^3

V_5: a linear section $\operatorname{Gr}(2,5) \cap P^6$ of the Grassmann $\operatorname{Gr}(2,5) \hookrightarrow P^9$ (Plücker embedding) by three hyperplanes in P^9, which is the Fano threefold of the index two, degree 5 in P^5

H_5^0: a normal hyperplane section of V_5 with exactly one rational double point of A_4-type, which is also the degenerated del-Pezzo surface of degree 5

H_5^∞: a non-normal hyperplane section of V_5 whose singular locus is a line Σ with the normal bundle $N_{\Sigma|V_5} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(1)$, in particular, H_5^∞ is a ruled surface swept out by lines in V_5 intersecting the line Σ

V_{22}: the Fano threefold of index one with the genus $g=12$, degree 22 in P^{13} (the anti-canonical embedding)

H_{22}^0 (resp. H_{22}^∞): a non-normal hyperplane section of V_{22} whose singular locus is a line Z with the normal bundle $N_{Z|V_{22}} \cong \mathcal{O}_Z(-2) \oplus \mathcal{O}_Z(1)$, and the multiplicity $\operatorname{mult}_Z H_{22}^0$ (resp. $\operatorname{mult}_Z H_{22}^\infty$) of H_{22}^0 (resp. H_{22}^∞) at a general point of Z is equal to two (resp. three), in particular, H_{22}^∞ is a ruled surface swept out by conics in V_{22} intersecting the line Z.

The proof of Theorem in the case of $r \geq 2$ was given in [2], [5], [8].

In the case of $r=1$, we have to look carefully at the structure of non-normal projective surfaces with the trivial dualizing sheaves and the double projection of V_{22} from a line or a conic. The details will be published elsewhere. Now, in this paper, we will show how these compactifications of $\boldsymbol{C}^3$ are constructed from the well-known compactification $\boldsymbol{P}^3$.

Construction. 1. Let L be a hyperplane in $\boldsymbol{P}^3$. Then one can see that $\boldsymbol{P}^3-L\cong\boldsymbol{C}^3$, and thus we have the compactification $(\boldsymbol{P}^3, L)$ of $\boldsymbol{C}^3$ of the index $r=4$.

2. Let $(\boldsymbol{P}^3, L)$ be as above. Let C be a conic in L and L' a hyperplane in $\boldsymbol{P}^3$ such that $C\cdot L'=2p$ (double point). Let $\lambda_C: B_c(\boldsymbol{P}^3)\rightarrow\boldsymbol{P}^3$ be the blowing up of $\boldsymbol{P}^3$ along C and put $C':=\lambda_C^{-1}(C)\cong\boldsymbol{F}_2$ (Hirzebruch surface). Let $\bar{L}$, $\bar{L}'$ be the proper transforms of L, L' respectively.

Then we have:

(2.1) There is a birational morphism $\pi_{\bar{L}}: B_C(\boldsymbol{P}^3)\rightarrow\boldsymbol{Q}^3$ of $B_C(\boldsymbol{P}^3)$ onto a smooth quadric hypersurface $\boldsymbol{Q}^3$ in $\boldsymbol{P}^4$, which contracts $\bar{L}\cong\boldsymbol{P}^2$ to a smooth point $v:=v_L=\pi_{\bar{L}}(\bar{L})$.

We put $\varphi_{(C,L)}: \pi_{\bar{L}}\circ\lambda_C^{-1}: \boldsymbol{P}^3\cdots\rightarrow\boldsymbol{Q}^3$, and $Q:=\varphi_{(C,L)}(C)=\pi_{\bar{L}}(C')$, $Q':=\varphi_{(C,L)}(L')=\pi_{\bar{L}}(\bar{L}')$, $g:=\varphi_{(C,L)}(p)=\pi_{\bar{L}}(\lambda_C^{-1}(p))$.

Then we have:

(2.2) $\varphi_{(C,L)}: \boldsymbol{P}^3-L\cong\boldsymbol{Q}^3-Q$ (isomorphic),

(2.3) Q, Q' are quadric cones in $\boldsymbol{P}^3$, and the vertex of Q is the point $v=v_L$,

(2.4) g is a generating line of Q, Q' with $Q\cdot Q'=2g$ (double line),

(2.5) $(\boldsymbol{Q}^3, Q)\cong(\boldsymbol{Q}^3, Q')$.

We put $Q:=\boldsymbol{Q}_0^2$ $(\cong Q')$. Then $(\boldsymbol{Q}^3, \boldsymbol{Q}_0^2)$ is the compactification of $\boldsymbol{C}^3$ of the index $r=3$.

3. Let $(\boldsymbol{Q}^3, Q)$, $(\boldsymbol{Q}^3, Q')$, g, v be as above. Let D be a twisted cubic curve in Q such that $D\cap Q'=D\cap g=\{v\}$. Such a D always exists (cf. [2]). Let $\lambda_D: B_D(\boldsymbol{Q}^3)\rightarrow\boldsymbol{Q}^3$ be the blowing up of $\boldsymbol{Q}^3$ along $D\cong\boldsymbol{P}^1$ and put $D':=\lambda_D^{-1}(D)\cong\boldsymbol{F}_3$. Let $\bar{Q}$, $\bar{Q}'$, $\bar{g}$ be the proper transforms of Q, Q', g in $B_D(\boldsymbol{Q}^3)$, respectively.

Then we have:

(3.1) There is a birational morphism $\pi_{\bar{Q}}: B_D(\boldsymbol{Q}^3)\rightarrow V_5$ of $B_D(\boldsymbol{Q}^3)$ onto a Fano threefold V_5 of the first kind with the index two, degree 5 in $\boldsymbol{P}^6$ (see Notation), which contracts the ruled surface $\bar{Q}\cong\boldsymbol{F}_2$ to a line $\Sigma:=\pi_{\bar{Q}}(\bar{Q})$ in V_5.

We put $\varphi_{(D,Q)}: \pi_{\bar{Q}}\circ\lambda_D^{-1}: \boldsymbol{Q}^3\cdots\rightarrow V_5\hookrightarrow\boldsymbol{P}^6$, and $H_5:=\varphi_{(D,Q)}(D)=\pi_{\bar{Q}}(D')$, $H_5':=\varphi_{(D,Q)}(Q')=\pi_{\bar{Q}}(\bar{Q}')$, $w:=w_g=\varphi_{(D,Q)}(g)=\pi_{\bar{Q}}(\bar{g})$ (a point of V_5).

Then we have:

(3.2) $\varphi_{(D,Q)}: \boldsymbol{Q}^3-Q\cong V_5-H_5$ (isomorphic),

(3.3) Σ is a line on V_5 with the normal bundle $N_{\Sigma|V_5}\cong\mathcal{O}_\Sigma(-1)\oplus\mathcal{O}_\Sigma(1)$,

(3.4) H_5 is a non-normal hyperplane section of V_5 whose singular locus is the line Σ, in particular, H_5 is a ruled surface swept out by lines intersecting the line Σ, (3.4)′ H_5' is a normal hyperplane section of V_5 with exactly one rational double point $w=w_g$ of A_4-type,

(3.5) $H_5 \cap H_5' = \Sigma$ (as a set), and $H_5H_5' = 5\Sigma$,

(3.6) $V_5 - H_5 \cong \boldsymbol{C}^3 \cong V_5 - H_5'$,

(3.7) (V_5, H_5), (V_5, H_5') are determined uniquely up to isomorphism (cf. [5]).

We put $H_5^{\infty} := H_5$, $H_5^0 := H_5'$ respectively. Then (V_5, H_5^{∞}) (V_5, H_5^0) are the compactification of $\boldsymbol{C}^3$ of the index $r=2$.

4. Let (V_5, H_5), (V_5, H_5'), Σ, $w=w_g$ be as above. Let E be a smooth rational curve of degree 5 in $H_5 \hookrightarrow V_5$ such that $E \cap H_5' = E \cap \Sigma = \{w\}$. Such an E always exists (cf. [4]). Let $\lambda_E: B_E(V_5) \to V_5$ be the blowing up of V_5 along $E \cong \boldsymbol{P}^1$ and put $E' := \lambda_E^{-1}(E)$. Let $\bar{H}_5$, $\bar{H}_5'$, $\bar{\Sigma}$ be the proper transforms of H_5, H_5', Σ in $B_E(V_5)$, respectively.

Then we have:

(4.1) There is a birational map, called a "flop", $\mu: B_E(V_5) \cdots\to U$ of $B_E(V_5)$ onto a smooth projective threefold U such that $\mu: B_E(V_5) - \bar{\Sigma} \cong U - \Delta$, where Δ is some smooth rational curve in U with the normal bundle $N_{\Delta|U} \cong \mathcal{O}_\Delta(-2) \oplus \mathcal{O}_\Delta$.

Let H, H', Z' be the proper transforms of E', $\bar{H}_5$, $\bar{H}'$ respectively. Then we have:

(4.2) There is a birational morphism $\pi_{Z'}: U \to V_{22}$ of U onto a Fano threefold V_{22} of the first kind with the index one, the genus $g=12$ (see Notation), which contracts the surface $Z' \cong \boldsymbol{F}_3$ to a line $Z := \pi_{Z'}(Z')$.

We put $\varphi_{(E,H_5)} := \pi_{Z'} \circ \mu \circ \lambda_E^{-1}: V_5 \cdots\to V_{22} \hookrightarrow \boldsymbol{P}^{13}$, and $H_{22} := \varphi_{(E,H_5)}(E) = \pi_{Z'}(H)$, $H_{22}' := \varphi_{(E,H_5)}(H_5') = \pi_{Z'}(H')$. In particular, $Z = \varphi_{(E,H_5)}(H_5)$. Then we have:

(4.3) $\varphi_{(E,H_5)}: V_5 - H_5 \cong V_{22} - H_{22}$ (isomorphic),

(4.4) Z is a line on V_{22} with the normal bundle $N_{\Delta|V_{22}} \cong \mathcal{O}_\Delta(-2) \oplus \mathcal{O}_\Delta(1)$,

(4.5) H_{22} is a non-normal hyperplane section of V_{22} whose singular locus is the line Z, and $\mathrm{mult}_Z H_{22} = 3$ (the multiplicity of H_{22} at a general point of Z), in particular, H_{22} is a ruled surface swept out by conics intersecting the line Z.

(4.5)′ H_{22}' is also a non-normal hyperplane section of V_{22} whose singular locus is the same line Z, and $\mathrm{mult}_Z H_{22}' = 2$,

(4.6) $V_{22} - H_{22} \cong \boldsymbol{C}^3 \cong V_{22} - H_{22}'$ (cf. [4]),

(4.7) (V_{22}, H_{22}), (V_{22}, H_{22}') are not determined uniquely up to isomorphism, they have a 4-dimensional family (cf. [7]).

We put $H_{22} ;= H_{22}^{\infty}$, $H_{22}' := H_{22}^0$, respectively. Then these $(V_{22}^{\infty}, H_{22})$, (V_{22}, H_{22}^0) are the compactifications of the index $r=1$.

Thus we have finally the following sequence of birational maps among the compactifications of $\boldsymbol{C}^3$:

$$\begin{array}{ccccccc}
(\boldsymbol{P}^3, L) & \overset{}{\underset{\varphi_{(C,L)}}{\cdots\cdots\to}} & (\boldsymbol{Q}^3, Q) & \underset{\varphi_{(D,Q)}}{\cdots\cdots\to} & (V_5, H_5) & \underset{\varphi_{(E,H_5)}}{\cdots\cdots\to} & (V_{22}, H_{22}) \\
\wr\| & & \wr\| & & & & \\
(\boldsymbol{P}^3, L') & \cdots\cdots\to & (\boldsymbol{Q}^3, Q') & \cdots\cdots\to & (V_5, H_5') & \cdots\cdots\to & (V_{22}, H_{22}')
\end{array}$$

Conclusion. Any smooth projective compactification of C^3 with the second Betti number equal to one can be obtained from the compactification (P^3, P^2) by the above way.

Acknowledgements. The author wishes to express his hearty thanks to Professor H. Grauert and Professor H. Flenner for giving him the opportunity of visiting SFB 170 "Geometrie und Analysis".

References

[1] L. Brenton and J. Morrow: Compactifications of C^n. Trans. Amer. Math. Soc., **246**, 139–158 (1979).
[2] M. Furushima: Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space C^3. Nagoya Math. J., **104**, 1–28 (1986).
[3] ——: Mukai-Umemura's example of a Fano threefold of genus 12 as a compactification of C^3 (1990) (preprint).
[4] ——: A new example of a compactification of C^3. Mathematicá Göttingensis, Heft 33 (1991).
[5] M. Furushima and N. Nakayama: The family of lines on the Fano threefold V_5. Nagoya Math. J., **116**, 111–122 (1989).
[6] V. A. Iskovskih: Anticanonical models of three-dimensional algebraic varieties. J. Soviet Math., **13-14**, 745–814 (1980).
[7] S. Mukai: On Fano threefolds (to appear in Projective Geometry, Trieste, 1989).
[8] Th. Peternell and M. Schneider: Compactifications of C^3. I. Math. Ann., **280**, 129–145 (1988).