78. *A Criterion for Multivalent Functions*

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Abstract: A more general criterion for multivalent functions is obtained. The result of this paper is the extension of the former results of Ozaki [1], Nunokawa [2], Nunokawa and Hoshino [3].

1. Introduction. It is well-known that if a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic and satisfies the condition $\operatorname{Re} f'(z) > 0$ in the unit disk $E = \{z : |z| < 1\}$, then $f(z)$ is univalent in E. Ozaki [1, Theorem 2] extended this result to the following:

If $f(z)$ is analytic in a convex domain D and $\operatorname{Re}(e^{i\alpha} f^{(p)}(z)) > 0$ in D, where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and $\operatorname{Re}(f^{(p)}(z)) > 0$ in E, then $f(z)$ is p-valent in E.

The above result was improved as follows:

Theorem A ([2]) . *Let $p \geqslant 2$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$\operatorname{Re} f^{(p)}(z) > -\frac{\log^{(4/e)}}{2\log(e/2)} p! \text{ in } E, \tag{1,1}$$

then $f(z)$ is p-valent in E.

Theorem B ([3]) . *Let $p \geqslant 3$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and*

$$\operatorname{Re} f^{(p)}(z) > -\frac{1 - 4\log(4/e)\log(e/2)}{4\log(4/e)\log(e/2)} p! \text{ in } E, \tag{1,2}$$

then $f(z)$ is p-valent in E.

In the present paper, we shall give a more general theorem which extends the above results.

2. Main Result. In order to derive our main result, we need the following lemmata.

Lemma 1 ([3]) . *Let $p(z)$ be analytic in E with $p(0) = 1$. Suppose that $\alpha > 0$, $\beta < 1$ and that for $z \in E$, $\operatorname{Re}(p(z) + \alpha z p'(z)) > \beta$. Then for $z \in E$,*

$$\operatorname{Re}(p(z)) > 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\alpha n}. \tag{2,1}$$

The estimate is best possible for

$$p_o(z) = 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\alpha n} z^n. \tag{2,2}$$

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Lemma 2 ([4, Theorem 8]) . *Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in E. If there exfists a $(p-k+1)$-valent starlike function $g(z) = z^{p-k+1} + \sum_{n=p-k+2}^{\infty} b_n z^n$ that satisfies*

$$\operatorname{Re}\frac{zf^{(k)}(z)}{g(z)} > 0 \qquad Z \in E, \tag{2,3}$$

then $f(z)$ is p-valent in E.

Theorem. *Let $p \geqslant m$, $(m \in \{2,3,4,\ldots\})$. If $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ is anlytic in E and*

$$\operatorname{Re} f^{(p)}(z) > -\left(\frac{(-1)^{m-1}}{2^{m-1}\prod_{k=1}^{m-1} A(1/k)} - 1\right)p! \quad \text{in } E, \tag{2,4}$$

where $A(1/k) = (-1)^k k\left(\log(1/2) - \sum_{n=1}^{K}\frac{(-1)^n}{n}\right)$. Then $f(z)$ is p-vaoent in E.

Proof. Let

$$\beta_o = 1 - \frac{(-1)^{m-1}}{2^{m-1}\prod_{k=1}^{m-1} A(1/k)},$$

and define $\beta_k (k = 1,2,\ldots)$ by

$$\beta_k = 1 + 2(1-\beta_{k-1})\,A\,(1/k).$$

One can show that

$$(-1)^{m-1}(1-\beta_{m-1}) = 2^{m-1}(1-\beta_0)\prod_{k=1}^{m-1} A(1/k)\,; \tag{2,5}$$

$$|A(1/k)| \leqslant \frac{k}{1+k} < 1,\ A\left(\frac{1}{1+k}\right) = -\left(1 + \frac{1+k}{k}A\,(1/k)\right). \tag{2,6}$$

So, $\beta_{m-1} = 0$ and for all $k = 1,2,3,\ldots$, $A(1/k) < 0$, $\beta_{K+1} < 1$.

Defining the function $p_k(z)$ by

$$p_k(z) = \frac{k!\,f^{(p-k)}(z)}{p!\,z^k},\ (k = 1,2,\ldots, m-1, z \in E).$$

Then $p_k(0) = 1$ and

$$\operatorname{Re}(p_k(z) + \frac{1}{k}zp_k'(z)) = \operatorname{Re}\left(\frac{(k-1)!\,f^{(p-k+1)}(z)}{p!\,z^{k-1}}\right. = Re\,p_{k-1}(z), \qquad (z \in E). \tag{2,7}$$

Applying Lemma 1, we can see that if $\operatorname{Re} p_{k-1}(z) > \beta_{k-1}$ in E, then $\operatorname{Re} p_k(z) > \beta_k$ in E, where β_k is defined above. Hence,

$$\operatorname{Re}\frac{f^{(p)}(z)}{p!} = \operatorname{Re} p_0(z) > \beta_0 \ \Rightarrow\ \operatorname{Re} p_{m-1}(z) > \beta_{m-1} = 0, (z \in E). \tag{2,8}$$

This shows that, under the hypothesis of the theorem, we have

$$\operatorname{Re}\frac{zf^{(p-m+1)}(z)}{z^m} > 0, \qquad (z \in E),$$

It is trivial that $g(z) = z^m$ is m-valently starlike in E. Therefore, from Lemma 2, we see that $f(z)$ is p-valent in E. The proof of the Theorem is complete.

Remark. If $m = 2$, then $\beta_0 = -0.62944\ldots.$
If $m = 3$, then $\beta_0 = -1.10907\ldots.$
If $m = 4$, then $\beta_0 = -1.5074\ldots.$

References

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