74. A Criterion for Algebraicity of Analytic Set Germs

By Shuzo IZUMI

Department of Mathematics, Kinki University

(Communicated by Heisuke HIRONAKA, M. J. A., Dec. 14, 1992)

In this paper we characterize the germs of algebraic subsets among the germs of analytic subsets by validity of an inequality between orders and degrees for polynomial functions on them.

Throughout this paper K denotes the field $\boldsymbol{C}$ or $\boldsymbol{R}$. Let S be a germ at 0 of an analytic subset of an open neighborhood of $0 \in K^n$ and $\nu_{S,0}(f)$ the vanishing order of $f \in K[x_1, \ldots, x_n]$ at 0 along S. To be accurate, if $\mathfrak{i}_0 \subset K\{x_1, \ldots, x_n\}$ is the analytic ideal of S at 0, $\mathfrak{m} \equiv (x_1, \ldots, x_n) \subset K\{x_1, \ldots, x_n\}$ the maximal ideal at 0 and if $f_0 \in K\{x_1, \ldots, x_n\}$ is the germ of f at 0, we put

$$\nu_{S,0}(f) = \max\{r \in \boldsymbol{N} : f_0 \in \mathfrak{m}^r + \mathfrak{i}_0\}.$$

Theorem. *Let S be a germ at 0 of an analytic subset of an open neighborhood of $0 \in K^n$. Suppose that S is irreducible and of positive dimension. Then the following conditions are equivalent.*

$(*)$ *S is an analytic irreducible component of the germ of an algebraic subset.*

$(**)$ *There exists $a \in \boldsymbol{R}$ such that $a \cdot \deg f \geqq \nu_{S,0}(f)$ for any $f \in K[x_1, \ldots, x_n]$ that does not vanish identically on S. Such an a must satisfy $a \geqq 1$.*

We may replace $\nu_{S,0}(f)$ in the above by the reduced order $\bar{\nu}_{S,0}(f) \equiv \lim_{k\to\infty} \nu_{S,0}(f^k)/k$ (cf. [3]). Our theorem exhibits an analogy to Sadullaev's theorem [4] which characterizes the algebraic subsets by a growth estimate of polynomial functions (cf. [1] for "analogy").

The complex case of $(*) \Rightarrow (**)$ is already known (a slight modification of [1], Thm. A*, (2.1), where the author has carelessly omitted the non-vanishing condition for f). The real case of $(*) \Rightarrow (**)$ easily follows from the complex case. Since $\nu_{S,0}(f) \geqq \deg f$ holds for homogeneous f which does not vanish identically on S, $a \geqq 1$ follows. Thus we have only to prove $(**) \Rightarrow (*)$.

Proof of $(**) \Rightarrow (*)$. Let $J \subset K[x_1, \ldots, x_n]$ be the ideal of all polynomials which vanish on S. J defines the minimal algebraic subset $\tilde{T} \subset K^n$ such that its germ T at 0 includes S. Let p and q denote the dimensions of S and T respectively. In complex case it is well-known that $\dim \tilde{T} = q$. In real case, the same equality follows from the following:

$$\dim_{\boldsymbol{R}} T = \dim_{\boldsymbol{C}} T^{\boldsymbol{C}} = \dim_{\boldsymbol{C}} (T^{\boldsymbol{C}})^{\sim} \geqq \dim_{\boldsymbol{R}} (T^{\boldsymbol{C}})^{\sim} \cap \boldsymbol{R}^n \geqq \dim_{\boldsymbol{R}} \tilde{T} \geqq \dim_{\boldsymbol{R}} T,$$

where $T^{\boldsymbol{C}}$ denotes the complexification of T and the first equality follows from [2], V, Prop.3. Let us put

$$A \equiv K[x_1, \ldots, x_n],\ A_k \equiv \{f \in A : \deg f \leqq k\},\ J_k \equiv J \cap A_k.$$

A_k/J_k is the set of the germs of polynomial functions on $\tilde{T}$ of degree $\leqq k$. We can naturally identify K^n with an affine chart of the projective space $\boldsymbol{KP}^n$. Then, by the theory of Hibert polynomial applied to the closure of

$\tilde{T} \subset \boldsymbol{KP}^n$, there exists a polynomial ϕ of degree q such that $\dim_K A_k / J_k = \phi(k)$ for $k \gg 0$. Let ψ denote the Samuel polynomial for the analytic local ring $\mathcal{O}_{S,0}$. It is known that $\deg \psi = p$.

Then we can find $c > 0$ such that $\psi(k^q) < \phi(ck^p)$ for $k \gg 0$ and hence the canonical K-linear map $A_{ck^p}/J_{ck^p} \to \mathcal{O}_{S,0}/\mathfrak{n}^{k^q}$ is not injective for $k \gg 0$, where $\mathfrak{n}$ denotes the maximal ideal of $\mathcal{O}_{S,0}$. Thus for any $k \gg 0$, there exists $f \in A_{ck^p} \backslash J_{ck^p}$ such that $f_0 \in \mathfrak{m}^{k^q} + \mathfrak{i}_0$ i.e. $\nu_{S,0}(f) = k^q$. Note that f does not vanish identically on S (otherwise, $f \in J \cap A_{ck^p} = J_{ck^p}$, a contradiction).

Suppose now that S is not an analytic irreducible component of T. Then S is properly included in some irreducible component T' of T and $p \equiv \dim_K S < \dim_K T' \leqq \dim_K T \equiv q$ by [2], Ⅲ, Prop.7. Therefore, for any $a > 0$, we have $a \cdot ck^p < k^q$ for $k \gg 0$. Then $(**)$ contradicts the existence of f above. This comletes the proof of $(**) \Rightarrow (*)$.

Let us put

$$\alpha(S, 0) \equiv \varlimsup_{k\to\infty} \sup\{\log \nu_{S,0}(f)/\log \deg f : f \in A_k \backslash J_k\}.$$

The inequality $\alpha(S, 0) \geqq 1$ holds by the same reason as $a \geqq 1$ above. If S is algebraic (i.e. S is an analytic irreducible component of an algebraic germ), the implication $(*) \Rightarrow (**)$ implies that $\alpha(S, 0) \leqq 1$. Therefore $\alpha(S, 0) = 1$. Conversely, Suppose that S is not algebraic. The proof above implies that $\alpha(S, 0) \geqq q/p > 1$. Thus $\alpha(S, 0)$ measures transcendency of S along with q/p.

Example 1. Let us define irreducible germ S of an analytic subset of K^2 by the equation $y = xe^x$. Then $J = 0$ and

$$1\,;\, x, y\,;\, x^2, xy, y^2\,;\ldots;\, x^k, x^{k-1}y, \ldots, y^k$$

form a K-basis of A_k. Their images in $\mathcal{O}_{S,0}$ are represented by

$$1\,;\, x, xe^x\,;\, x^2, x^2e^x, x^2e^{2x}\,;\ldots;\, x^k, x^ke^x, x^ke^{2x}, \ldots, x^ke^{kx} \in K\{x, y\}.$$

These are independent solutions of the ordinary differential equation $\{D(D-1)(D-2)\cdots(D-k)\}^{k+1} f = 0$. Let f be a linear combination of these solutions. If $\nu_{S,0}(f) \geqq (k+1)^2$ (= the order of the differential equation), $f \equiv 0$ by the uniqueness of solution. In other words, $f \in A_k \backslash \{0\} = A_k \backslash J_k$ implies $\nu_{S,0}(f) < (k+1)^2$. This proves that $\alpha(S, 0) \leqq 2$. Since $\alpha(S, 0) \geqq q/p = 2$ follows from the above, we have $\alpha(S, 0) = 2$.

Example 2 (T. Ueda). Let $n_1, n_2, n_3, \ldots \in \boldsymbol{N}$ be an increasing series such that $\log n_{i+1}/\log n_i \to \infty$ (e.g. $n_i = 2^{i!}$) and $c_1, c_2, c_3, \ldots \in K$ a series such that the radius of convergence of $f(x) = \sum_{i=1}^{\infty} c_i x^{n_i}$ is infinite (e.g. $c_i = 1/n_i!$). Let us put

$$\tilde{S} = \{(x, y) : y = f(x)\},\ f_k(x) = \sum_{i=1}^{k} c_i x^{n_i}.$$

(f is an entire function and $\tilde{S}$ is a closed analytic subset of K^n.) Then $\deg(y - f_k) = n_k$ and $\nu_{S,0}(y - f_k) = n_{k+1}$ and we have

$$\alpha(S, 0) = \lim \log n_{k+1}/\log n_k = \infty > 2 = q/p.$$

Thus $\alpha(S, 0)$ is more sensitive than q/p as a measure of transcendency. The author does not know when $\alpha(S, 0)$ is finite.

Let $\tilde{S}$ be a closed irreducible analytic subset of K^n. The author does not know whether $\alpha(\tilde{S}_P^i, P)$ (defined in an obvious manner) is always independent of $P \in \tilde{S}$ and local irreducible component $\tilde{S}_P^i$.

Acknowledgements. The author thanks Professor T. Ueda and Professor M. Shiota, for invaluable suggestions.

References

[1] S. Izumi: Increase, convergence and vanishing of functions along a Moishezon space. J. Math. Kyoto Univ., **32**, 245–258 (1992).

[2] R. Narasimhan: Introduction to the theory of analytic spaces (LNM 25). Springer, Berlin (1966).

[3] D. Rees: Valuations associated with a local ring. II. J. London Math. Soc., **31**, 228–235 (1956).

[4] A. Sadullaev: An estimate for polynomials on analytic sets. Math. USSR Izv., **20**, 493–502 (1983).