71. The Generalized Confluent Hypergeometric Functions[†)]

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Introduction. The purpose of this note is to introduce a class of hypergeometric functions of confluent type defined on the Grassmannian manifold $G_{r,n}$, the moduli space for r-dimensional linear subspace in $\boldsymbol{C}^n$. These functions will be called the generalized confluent hypergeometric functions.

Let r and $n(n>r)$ be positive integers and let $Z_{r,n}$ be the set of $r\times n$ complex matrices of maximal rank. On $Z_{r,n}$ there are natural actions of $GL(r, \boldsymbol{C})$ and of $GL(n, \boldsymbol{C})$ by the left and right matrix multiplications, respectively, and the Grassmannian manifold $G_{r,n}$ is identified with the space $GL(r, \boldsymbol{C})\backslash Z_{r,n}$. Let $\psi: Z_{r,n}\to G_{r,n}$ be the natural projection map. In Section 1, we define the system of partial differential equations on $Z_{r,n}$ which will be called the generalized confluent hypergeometric system. This system induces the system on $G_{r,n}$ through the mapping ψ (see Section 1).

There is given a partition of n, $\lambda=(\lambda_1,\dots,\lambda_l)$, i.e. the sequence of positive integers $\lambda_1\geq\cdots\geq\lambda_l>0$ satisfying $|\lambda|=\lambda_1+\cdots+\lambda_l=n$. For a partition λ, we define the maximal commutative subgroup H_λ of $GL(n, \boldsymbol{C})$ (see the definition in Section 1) which acts on $Z_{r,n}$ as a subgroup of $GL(n, \boldsymbol{C})$. Our generalized confluent hypergeometric functions $F(z)$ on $Z_{r,n}$ will be a multi-valued analytic function satisfying the homogeneity property:

$$(1)\qquad \begin{cases} F(zc)=F(z)\chi_\alpha(c) & \text{for } c\in H_\lambda, \\ F(gz)=(\det g)^{-1}F(z) & \text{for } g\in GL(r, \boldsymbol{C}), \end{cases}$$

where χ_α is a character of the universal covering group of H_λ (see Section 1). This property implies that the functions $F(z)$ in $Z_{r,n}$ induces multi-valued functions on the quotient space $X_\lambda := G_{r,n}/H_\lambda$. In the case $\lambda=(1,\dots,1)$, the confluent hypergeometric function $F(z)$ coincides with the general hypergeometric function of I. M. Gelfand [1] and in the case $\lambda=(n)$, it coincides with the generalized Airy function due to Gelfand, Retahk and Serganova [3].

1. Generalized confluent hypergeometric functions. The Jordan group $J(m)$ of size m is a commutative subgroup of $GL(m, \boldsymbol{C})$ defined by

$$J(m) := \left\{c=\sum_{i=0}^{m-1} c_i\tau^i\,;\ c_i\in\boldsymbol{C},\ c_0\neq 0\right\},$$

where

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$$\tau = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in M(m, \boldsymbol{C})$$

Note that this group is the centralizer of an element τ. In what follows we denote by $[c_0, c_1, \ldots, c_{m-1}]$ an element $\sum_{i=0}^{m-1} c_i \tau^i$. The subgroup $J_0(m)$ of the Jordan group $J(m)$ is defined by

$$J_0(m) := \{[1, c_1, \ldots, c_{m-1}] \in M(m, \boldsymbol{C})\ ;\ c_1, \ldots, c_{m-1} \in \boldsymbol{C}\}.$$

Then we have

Proposition 1. *There are following group isomorphisms.*
(i) $J(m) \simeq \boldsymbol{C}^{\times} \times J_0(m)$,
(ii) $J_0(m) \simeq \boldsymbol{C}^{m-1}$, *where* $\boldsymbol{C}^{m-1}$ *is equipped with the natural additive structure.*

The isomorphism $J(m) \simeq \boldsymbol{C}^{\times} \times J_0(m)$ is given by the correspondence $[c_0, c_1, \ldots, c_{m-1}] \mapsto (c_0, [1, c_1/c_0, \ldots, c_{m-1}/c_0])$. The isomorphism $J_0(m) \to \boldsymbol{C}^{m-1}$ is constructed as follows: Let $b = 1 + b_1 T + b_2 T^2 + \cdots \in \boldsymbol{C}[[T]]$ be a formal power series in the indeterminate T. We define the polynomials $\theta_i(b_1, \ldots, b_i)$, $(i = 1, 2, \ldots)$, by

$$\log b = \sum_{i=1}^{\infty} \theta_i(b_1, \ldots, b_i) T^i.$$

If we define the weight of b_j to be equal to j the polynomials θ_i are weighted homogeneous of weight i. For example, we have

$$\theta_1 = b_1, \qquad \theta_2 = b_2 - \frac{1}{2} b_1^2,$$

$$\theta_3 = b_3 - b_1 b_2 + \frac{1}{3} b_1^3, \quad \theta_4 = b_4 - \frac{1}{2}(2b_1 b_3 + b_2^2) + b_1^2 b_2 - \frac{1}{4} b^4.$$

By the definition of θ_i, for $c = [1, c_1, c_2, \ldots, c_{m-1}] \in J_0(m)$, we have

$$\log c = \sum_{i=1}^{m-1} \theta_i(c_1, \ldots, c_i) \tau^i.$$

Then the correspondence $J_0(m) \ni c = [1, c_1, \ldots, c_{m-1}] \mapsto (\theta_1(c), \ldots, \theta_{m-1}(c))$ gives the desired isomorphism.

Next we give the characters of the universal covering group $\tilde{J}(m)$ of Jordan group $J(m)$, i.e. the multiplicative homomorphism $\chi : \tilde{J}(m) \to \boldsymbol{C}^{\times}$: $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. In what follows, we say "the character of $J(m)$" instead of saying "the character of $\tilde{J}(m)$", by the abuse of language.

Proposition 2. *A character* χ *of the group* $J(m)$ *is given by*

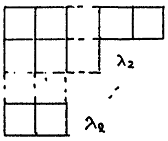
$$\chi([c_0, c_1, \cdots, c_{m-1}]) = c_0^{\alpha_0} \exp\Big(\sum_{i=1}^{m-1} \alpha_i \theta_i(c_1/c_0, \cdots, c_i/c_0)\Big),$$

for some $\alpha_0, \cdots, \alpha_{m-1} \in \boldsymbol{C}$.

The character χ in the above proposition will be called the character of homogeneity $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$ and is denoted by χ_α in order to indicate the dependence on α.

Given a partition $\lambda = (\lambda_1, \cdots, \lambda_l)$ of n, in another term, a Young diagram λ of weight n consisting of boxes placed at integral points $(i, j) \in \boldsymbol{Z}^2$, $1 \leqslant i \leqslant l$, $0 \leqslant j \leqslant \lambda_i - 1$.

Fig. 1



With the Young diagram λ, we associate the maximal commutative subgroup $H_\lambda := J(\lambda_1) \times J(\lambda_2) \times \cdots \times J(\lambda_l)$ of the general linear group $GL(n, \boldsymbol{C})$. Note that any maximal commutative subgroup H is conjugate to H_λ for some Young diagram, i.e. $H_\lambda = Ad_g(H)$ for some $g \in GL(n, \boldsymbol{C})$.

Let $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(l)})$, $\alpha^{(i)} = (\alpha_0^{(i)}, \alpha_1^{(i)}, \ldots, \alpha_{\lambda_i-1}^{(i)})$ $(i = 1, \ldots, l)$, be an n-tuple of complex numbers satisfying

$$\sum_{i=1}^{l} \alpha_0^{(i)} = -r.$$

We define the character $\chi_\alpha : H_\lambda \to \boldsymbol{C}^\times$ by $\chi_\alpha(c) = \Pi_i \chi_{\alpha^{(i)}}(c^{(i)})$, where

$$c = (c^{(1)}, \cdots c^{(l)}) \in H_\lambda,\ c^{(i)} = [c_0^{(i)}, \cdots, c_{\lambda_i-1}^{(i)}] \in J(\lambda_i)$$

and $\chi_{\alpha^{(i)}}$ is the character of the group $J(\lambda_i)$ given in Proposition 2. We say that χ_α is the character of H_λ of homogeneity $\alpha \in \boldsymbol{C}^n$.

We want to obtain multi-valued function F on $Z_{r,n}$ satisfying the poperties (1). The first and the second property in (1) are called the H_λ-homogeneity and the $GL(r, \boldsymbol{C})$-homogeneity for $F(z)$, respectively.

Definition 3. *Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a Young diagram of weight n. The generalized confluent hypergeometric system* (*CHG system for short*) *of type λ is the system of partial differential equations :*

$$M_\lambda : \quad \begin{cases} L_{im}\, u = \alpha_m^{(i)} u, & i = 1, \ldots, l;\ m = 0, \ldots, \lambda_i - 1 \\ M_{ij}\, u = -\delta_{ij} u, & i, j = 1, \ldots, r \\ \square_{ij,pq}\, u = 0 & i, j = 1, \ldots, r;\ p, q = 1, \ldots, n \end{cases}$$

where

$$L_{im} = \sum_{q=1}^{r} \sum_{\substack{\lambda_1 + \cdots + \lambda_{i-1} + m + 1 \\ \leq p \leq \lambda_1 + \cdots + \lambda_i}} z_{q,p-m} \frac{\partial}{\partial z_{qp}},$$

$$M_{ij} = \sum_{p=1}^{n} z_{ip} \frac{\partial}{\partial z_{jp}}$$

$$\square_{ij,pq} = \frac{\partial^2}{\partial z_{ip} \partial z_{jq}} - \frac{\partial^2}{\partial z_{iq} \partial z_{jp}}.$$

Any solution of the system M_λ is called *the generalized confluent hypergeometric function* (CHGF for sort) of type λ.

It is easily seen that the equations $L_{im}\, u = \alpha_m^{(i)} u$ and $M_{ij}\, u = -\delta_{ij} u$ of the system M_λ are the infinitesimal expressions for the H_λ-homogeneity and the $GL(r, \boldsymbol{C})$-homogeneity of solutions for the system M_λ, respectively. This fact immediately leads to the

Proposition 4. *Any solution of the generalized confluent hypergeometric system M_λ satisfies the homogeneity property* (1).

The following theorem asserts that the system M_λ is like an ordinary differential equation, that is, in a neighbourhood of a "generic" point $a \in Z_{r,n}$, the space of solutions for M_λ is isomorphic to a finite dimensional complex vector space.

Theorem 5. *The generalized confluent hypergeometric system M_λ is holonomic.*

For a fixed Young diagram $\lambda=(\lambda_1,\dots,\lambda_l)$, we consider subsets $\mu \subset \lambda$, which are called the subdiagrams of λ, consisting of (i,j)-boxes such that $0 \le j \le \mu_i - 1$ $(i=1,\dots,l)$. They are denoted as $\mu=(\mu_1,\dots,\mu_l)$. The number of boxes $|\mu|$ in μ will be called the weight of $\mu : |\mu| = \mu_1 + \cdots + \mu_l$. Let the column vectors of $z \in Z_{r,n}$ be indexed as

(2) $$z=(z^{(1)},\dots,z^{(l)}),\ z^{(i)}=(z_0^{(i)},\dots,z_{\lambda_i-1}^{(i)}) \in M(r,\lambda_i).$$

For $z \in Z_{r,n}$ and a subdiagram $\mu \subset \lambda$ of weight r, we set $z_\mu=(z_0^{(1)},\dots,z_{\mu_1-1}^{(1)},\dots,z_0^{(l)},\dots,z_{\mu_l-1}^{(l)})$. Then a point $z \in Z_{r,n}$ is said to be *generic* with respect to H_λ if, for any subdiagram $\mu \subset \lambda$ of weight r, the $r \times r$ minor determinant $\det z_\mu$ does not vanish. Since this property is invariant by the action of $GL(r, C)$ and H_λ, we can say that a point of $G_{r,n}$ or that of $X_\lambda := G_{r,n}/H_\lambda$ is *generic* with respect to H_λ. The set of generic points of $Z_{r,n}$ (resp. $G_{r,n}$, X_λ) with respect to H_λ is called the *generic stratum* of $Z_{r,n}$ (resp. $G_{r,n}$, X_λ).

A point $z \in Z_{r,n}$ of the generic stratum with respect to H_λ can be normalized to an element $\zeta \in Z_{r,n}$ by the actions of $GL(r, C)$ and H_λ so that the variable entries of ζ furnishes local coordinates of the generic stratum of X_λ.

Hereafter, when $z \in Z_{r,n}$ is taken into $z' \in Z_{r,n}$ by the actions of $GL(r, C)$ and of H_λ, we write $z \sim z'$.

Let $\phi_* M_\lambda$ be the holonomic system on $G_{r,n}$ obtained from M_λ by the mapping $\phi : Z_{r,n} \to G_{r,n}$.

Proposition 6. *The system $\phi_* M_\lambda$ on the generic stratum in $G_{r,n}$ has at most $\binom{n-2}{r-1}$ linearly independent solutions.*

2. Integral representation. We study solution of the generalized confluent hypergeometric system M_λ given in an integral representation.

For $t=(t_1,\dots,t_r) \in C^r$ and for $z=(z_1,\dots,z_n) \in Z_{r,n}$, we set

$$\langle t, z\rangle = (\langle t, z_1\rangle,\dots,\langle t, z_n\rangle),$$

where

$$\langle t, z_j\rangle = \sum_{i=1}^{r} t_i z_{ij} \quad (1 \le j \le n).$$

Let ω be the $(r-1)$-form defined by

$$\omega := \sum_{i=1}^{r} (-1)^{i+1} t_i dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots dt_r.$$

Proposition 7. *For an appropriate $(r-1)$-dimensional chain Δ, the integral*

$$\Phi_\lambda(\alpha ; z) = \int_\Delta \chi_\alpha(\langle t, z\rangle) w$$

gives a solution of the CHG system M_λ.

3. Finite group symmetries. We give the finite group symmetries of the CHGF $\Phi_\lambda(\alpha ; z)$ which arises from the permutations of the rows with the same length of the Young diagram $\lambda=(\lambda_1,\dots,\lambda_l)$. To state the result precisely we introduce the following notation.

Suppose that $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies

$$\lambda_1 = \cdots = \lambda_{l_1} > \lambda_{l_1+1} = \cdots = \lambda_{l_2} > \cdots > \lambda_{l_{m-1}+1} = \cdots = \lambda_{l_m}, \ (l_m = l).$$

Let the column vectors of $z \in Z_{r,n}$ be indexed as in (2). Let S_i be the subgroup of symmetric group $\mathfrak{S}_l$ on l letters consisting of elements which permute the letters $l_{i-1}+1, \dots, l_i$ and leaves the other letters unchanged. Then $\sigma = (\sigma_1, \dots, \sigma_m) \in S_1 \times \cdots \times S_m \subset \mathfrak{S}_l$ acts on parameters of homogeneity $\alpha = (\alpha^{(1)}, \dots, \alpha^{(l)}) \in \boldsymbol{C}^n$ and on $z = (z^{(1)}, \dots, z^{(l)}) \in Z_{r,n}$ by the rule

$$\sigma_i \cdot \alpha = (\alpha^{(1)}, \dots, \alpha^{(l_{i-1})}, \alpha^{(\sigma_i(l_{i-1}+1))}, \dots, \alpha^{(\sigma_i(l_i))}, \alpha^{(l_i+1)}, \dots, \alpha^{(l)}),$$

$$\sigma_i \cdot z = (z^{(1)}, \dots, z^{(l_{i-1})}, z^{(\sigma_i(l_{i-1}+1))}, \dots, z^{(\sigma_i(l_i))}, z^{(l_i+1)}, \dots, z^{(l)}).$$

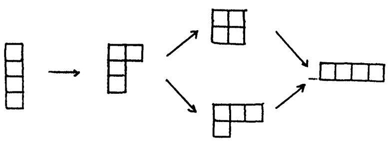
Then we have the following symmetries of the generalized confluent hypergeometric functions $\Phi_\lambda(\alpha\,;z)$.

Proposition 8. *We have*

$\Phi_\lambda(\sigma\cdot\alpha\,;\sigma\cdot z) = \Phi_\lambda(\alpha\,;z)$ *for any* $\sigma = (\sigma_1, \dots, \sigma_m) \in S_1 \times \cdots \times S_m$.

4. CHG functions and the classical confluent hypergeometric functions. In this paragraph we study the relation between the generalized confluent hypergeometric functions on $Z_{2,4}$ and the classical hypergeometric functions of confluent type obtained from the Gauss hypergeometric function by confluences. As in Section 1, we consider the Young diagrams λ of weight 4; that is $|\lambda| = 4$. They are tabulated as

(3) Fig. 2



Here if the two Young diagrams λ and μ are connected by an arrow as $\lambda \to \mu$ in the above diagram, it implies that μ is obtained from λ by making two rows of λ into a single row. With each Young diagram λ, $|\lambda| = 4$, the commutative subgroup H_λ of $GL(4, \boldsymbol{C})$ is associated. Then for a 4-tuple of complex constants $\alpha = (\alpha_1, \dots, \alpha_4)$, we consider the functions

$$\Phi_\lambda(\alpha\,;z) = \int_\Delta \chi_\alpha(\langle t, z\rangle)\omega, \quad z \in Z_{2,4},$$

where $\omega = t_1dt_2 - t_2dt_1$ and $\chi_\alpha(c)$ denotes the character of H_λ of homogeneity α. For $z = (z_1, \dots, z_4) \in Z_{2,4}$, we denote by $v_{ij} = \det(z_i, z_j)$ the (i, j)-th Plücker coordinate.

Proposition 9. *In the generic stratum of* $Z_{2,4}$, *the generalized confluent hypergeometric functions* $\Phi_\lambda(\alpha\,;z)$ *are related with functions on* X_λ *as*

$$\Phi_\lambda(\alpha\,;z) = C_\lambda(\alpha\,;z)\Phi_\lambda(\alpha\,;\zeta), \quad z \sim \zeta,$$

where $C_\lambda(\alpha\,;z)$ *is a product of powers of* v_{ij} *and exponential functions of* v_{ij}, *the variable entry* x *of* ζ *gives a coordinate in a chart of the generic stratum of* X_λ *and* $\Phi_\lambda(\alpha\,;\zeta)$ *is given by*

$$\Phi_{(1111)}(\alpha\,;\zeta) = \int u^{\alpha_2}(1-u)^{\alpha_3}(1-xu)^{\alpha_4}du, \qquad \zeta = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{pmatrix}$$

$$\Phi_{(211)}(\alpha\,;\,\zeta)=\int u^{-\alpha_1-\alpha_3-2}(1-u)^{\alpha_3}\exp(\alpha_2 xu)\,du,\quad \zeta=\begin{pmatrix}1&0&1&0\\1&x&0&1\end{pmatrix}$$

$$\Phi_{(22)}(\alpha\,;\,\zeta)=\int u^{-\alpha_3}\exp\Bigl(\alpha_2 u+\frac{x}{u}\Bigr)du,\quad \zeta=\begin{pmatrix}1&0&0&x\\0&1&1&0\end{pmatrix}$$

$$\Phi_{(31)}(\alpha\,;\,\zeta)=\int u^{-\alpha_4}\exp\Bigl(\alpha_2 u+\alpha_3\Bigl(-\frac{1}{2}u^2+xu\Bigr)\Bigr)du,$$
$$\zeta=\begin{pmatrix}0&1&x&1\\1&0&0&0\end{pmatrix}$$

$$\Phi_{(4)}(\alpha\,;\,\zeta)=\int\exp\Bigl(\alpha_2 u-\frac{1}{2}\alpha_3 u^2+\alpha_4\Bigl(\frac{1}{3}u^3+xu\Bigr)\Bigr)du,$$
$$\zeta=\begin{pmatrix}1&0&0&0\\0&1&0&x\end{pmatrix}.$$

The functions $\Phi_\lambda(\alpha,\zeta)$ reduce to classical hypergeometric functions of confluent type:

$\Phi_{(1111)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4\,;\,\zeta)=C\cdot F(\alpha_2,-\alpha_4,\alpha_2+\alpha_3+2\,;\,x)$: *Gauss hypergeometric function*

$\Phi_{(211)}(\alpha_1,\alpha_2,\alpha_3,1\,;\,\zeta)=C\cdot{}_1F_1(-\alpha_1,-\alpha_4,-1,-\alpha_1\,;\,x)$: *Kummer's confluent function*

$\Phi_{(22)}(\alpha_1,1,\alpha_3,-1\,;\,\zeta)=C\cdot x^{(\alpha_3+1)/2}J_{-\alpha_3-1}(\sqrt{4x})$: *Bessel function*

$\Phi_{(31)}(\alpha_1,0,1,\alpha_4\,;\,\zeta)=C\cdot H_{-\alpha_4-1}(x)$: *Hermite function*

$\Phi_{(4)}(\alpha_1,0,0,1\,;\,\zeta)=Ai(-x)$: *Airy function.*

Remark. The arrows in the diagram (3) correspond to the process of confluence for the differential equations M_λ. In fact, in [6] the hypergeometric functions $\Phi_\lambda(\alpha\,;\,\zeta)$ are obtained as particular solutions of Painlevé equations, which are also indexed by the Young diagrams of weight 4 from the point of view of monodromy preserving deformation and for Painlevé equations the arrows in (3) really implies the confluences.

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