8. *Certain Differential Operators for Meromorphically p-valent Convex Functions*

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(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1992)

Abstract: Let $J_n(\alpha)$ be the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty} a_k z^k \quad (a_{-p}\neq 0, p\in N=\{1,2,\cdots\})$$

which are regular in the punctured disk $E=\{z: 0<|z|<1\}$ and satisfying

$$\mathrm{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}-(p+1)\right\}<-p\frac{n+\alpha}{n+1} \quad (n\in N_0=\{0,1,2,\cdots\}, |z|<1, 0\leq\alpha<1),$$

where

$$D^n f(z)=\frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^n a_{m-1}z^{m-1}.$$

It is proved that $J_{n+1}(\alpha)\subset J_n(\alpha)$. Since $J_0(\alpha)$ is the class of meromorphically p-valent convex functions of order α, all functions in $J_n(\alpha)$ are p-valent convex. Futher properties preserving integrals are considered.

1. Introduction. Let $\sum_p$ denote the class of functions of the form

$$f(z)=\frac{a_{-p}}{z^p}+\sum_{k=0}^{\infty} a_k z^k \quad (a_{-p}\neq 0, p\in N=\{1,2,\cdots\}) \tag{1.1}$$

which are regular in the punctured disk $E=\{z: 0<|z|<1\}$. Define

$$D^0 f(z)=f(z), \tag{1.2}$$

$$D^1 f(z)=\frac{a_{-p}}{z^p}+(p+1)a_0+(p+2)a_1 z+(p+3)a_2 z^2+\cdots =\frac{(z^{p+1}f(z))'}{z^p}, \tag{1.3}$$

$$D^2 f(z)=D(D^1 f(z)), \tag{1.4}$$

and for $n=1,2,\cdots$,

$$D^n f(z)=D(D^{n-1}f(z))=\frac{a_{-p}}{z^p}+\sum_{m=1}^{\infty}(p+m)^n a_{m-1}z^{m-1} =\frac{(z^{p+1}D^{n-1}f(z))'}{z^p}. \tag{1.5}$$

In this paper, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$\mathrm{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}-(p+1)\right\}<-p\frac{n+\alpha}{n+1}, \quad (z\in U=\{z: |z|<1\}), \tag{1.6}$$

1991 Mathematics Subject classification: Primary 30C45.

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for some α $(0\leq\alpha<1)$ and $n\in N_0=\{0,1,2,\cdots\}$, is meromorphically p-valent convex in E. More precisely, it is proved that, for the classes $J_n(\alpha)$ of functions in $\sum_p$ satisfying (1.6),

$$J_{n+1}(\alpha)\subset J_n(\alpha) \tag{1.7}$$

holds. Since $J_0(\alpha)$ equals $\sum_k(\alpha)$ (the class of meromorphically p-valent convex functions of order α [4]), the convexity of members of $J_n(\alpha)$ is a consequence of (1.7). Further for $c>0$, let

$$F(z)=\frac{c}{c+p}\int_0^z t^{c+p-1}f(t)dt, \tag{1.8}$$

it is shown that $F(z)\in J_n(\alpha)$ whenever $f(z)\in J_n(\alpha)$. Some known results of Bajpai [1], Goel and Sohi [2] and Uralegaddi and Somanatha [6] are extended.

2. Properties of the class $J_n(\alpha)$. In proving our main results (Theorem 1 and Theorem 2 below), we shall need the following lemma due to I. S. Jack [3].

Lemma. *Let w be non-constant regular in $U=\{z:|z|<1\}$, $w(0)=0$. If $|w|$ attains its maximum value on the circle $|z|=r<1$ at z_0, we have $z_0w'(z_0)=kw(z_0)$ where k is a real number, $k\geq1$.*

Theorem 1. *$J_{n+1}(\alpha)\subset J_n(\alpha)$ for each integer $n\in N_0$.*

Proof. Let $f(z)\in J_{n+1}(\alpha)$. Then

$$\mathrm{Re}\left\{\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-(p+1)\right\}<-p\frac{n+1+\alpha}{n+2}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}-(p+1)\right\}<-p\frac{n+\alpha}{n+1}. \tag{2.2}$$

Define $w(z)$ in $U=\{z:|z|<1\}$ by

$$\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}-(p+1)=-p\left[\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\right]. \tag{2.3}$$

Clearly $w(z)$ is regular and $w(0)=0$. Equation (2.3) may be written as

$$\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}=\frac{n+1+(n+1+2p(1-\alpha))w(z)}{(n+1)(1+w(z))}. \tag{2.4}$$

Logarithmic differentiation of (2.4) yields

$$\frac{z(D^{n+1}f(z))''}{(D^{n+1}f(z))'}-\frac{z(D^{n}f(z))''}{(D^{n}f(z))'}=\frac{2p(1-\alpha)zw'(z)}{(1+w(z))(n+1+(n+1+2p(1-\alpha))w(z))}. \tag{2.5}$$

From the following identity, which is obvious from (1.5),

$$z(D^nf(z))'=D^{n+1}f(z)-(p+1)D^nf(z), \tag{2.6}$$

we obtain

$$z(D^nf(z))''=(D^{n+1}f(z))'-(p+2)(D^nf(z))'. \tag{2.7}$$

Using the identity (2.7), the equation (2.5) reduces to

$$\frac{((D^{n+2}f(z))')/((D^{n+1}f(z))')-(p+1)+p(n+1+\alpha)/(n+2)}{(1-\alpha)/(n+2)} \tag{2.8}$$

$$=p\left[\frac{1}{n+1}-\frac{(n+2)(1-w(z))}{(n+1)(1+w(z))}+\frac{2(n+2)zw'(z)}{(1+w(z))(n+1+(n+1+2p(1-\alpha))w(z))}\right].$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

(2.9) $$z_0 w'(z_0)=kw(z_0)$$

where $|w(z_0)|=1$ and $k\geq 1$. From (2.8) and (2.9), we obtain

(2.10) $$\frac{((D^{n+2}f(z_0))')/((D^{n+1}f(z_0))')-(p+1)+p(n+1+\alpha)/(n+2)}{(1-\alpha)/(n+2)}$$
$$=p\left[\frac{1}{n+1}-\frac{(n+2)(1-w(z_0))}{(n+1)(1+w(z_0))}+\frac{2k(n+2)w(z_0)}{(1+w(z_0))(n+1+(n+1+2p(1-\alpha))w(z_0))}\right].$$

Thus

(2.11) $$\mathrm{Re}\left\{\frac{((D^{n+2}f(z_0))')/((D^{n+1}f(z_0))')-(p+1)+p(n+1+\alpha)/(n+2)}{(1-\alpha)/(n+2)}\right\}$$
$$\geq p\left[\frac{1}{n+1}+\frac{n+2}{2(n+1+p(1-\alpha))}\right]>0,$$

which contradicts (2.1). Hence $|w(z)|<1$ in U and from (2.3) it follows that $f(z)\in J_n(\alpha)$.

Theorem 2. *Let $f(z)\in\sum_p$ satisfy the condition*

(2.12) $$\mathrm{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}-(p+1)\right\}<p\left[-\frac{n+\alpha}{n+1}+\frac{1-\alpha}{2(c(n+1)+p(1-\alpha))}\right]\quad(z\in U)$$

for a given $n\in N_0$ and $c>0$. Then

(2.13) $$F(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt$$

belongs to $J_n(\alpha)$.

Proof. From the definition of $F(z)$, we have

(2.14) $$z(D^n F(z))'=cD^n f(z)-(c+p)D^n F(z).$$

Using (2.14) and the identity (2.6), the condition (2.12) may be written as

(2.15) $$\mathrm{Re}\left\{\frac{((D^{n+2}F(z))')/((D^{n+1}F(z))')+(c-1)}{1+(c-1)((D^n F(z))')/((D^{n+1}F(z))')}-(p+1)\right\}$$
$$<p\left[-\frac{n+\alpha}{n+1}+\frac{1-\alpha}{2(c(n+1)+p(1-\alpha))}\right].$$

We have to prove that (2.15) implies the inequality

(2.16) $$\mathrm{Re}\left\{\frac{(D^{n+1}F(z))'}{(D^n F(z))'}-(p+1)\right\}<-p\frac{n+\alpha}{n+1}.$$

Define $w(z)$ in U by

(2.17) $$\frac{(D^{n+1}F(z))'}{(D^n F(z))'}-(p+1)=-p\left[\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\right].$$

Clearly $w(z)$ is regular and $w(0)=0$. The equation (2.17) may be written as

(2.18) $$\frac{(D^{n+1}F(z))'}{(D^n F(z))'}=\frac{n+1+(n+1+2p(1-\alpha))w(z)}{(n+1)(1+w(z))}.$$

Differentiating (2.18) logarithmically and using (2.7), we obtain

(2.19) $$\frac{(D^{n+2}F(z))'}{(D^{n+1}F(z))'}-\frac{(D^{n+1}F(z))'}{(D^n F(z))'}=\frac{2p(1-\alpha)zw'(z)}{(1+w(z))(n+1+(n+1+2p(1-\alpha))w(z))}.$$

The above equation may be written as

$$(2.20)\quad \frac{((D^{n+2}F(z))')/((D^{n+1}F(z))')+(c-1)}{1+(c-1)((D^{n}F(z))')/((D^{n+1}F(z))')}-(p+1)$$
$$=\frac{(D^{n+1}F(z))'}{(D^{n}F(z))'}-(p+1)+\left[\frac{2p(1-\alpha)zw'(z)}{(1+w(z))(n+1+(n+1+2p(1-\alpha))w(z))}\right]$$
$$\times\left[\frac{1}{1+(c-1)((D^{n}F(z))')/((D^{n+1}F(z))')}\right],$$

which, by using (2.17) and (2.18), reduces to

$$(2.21)\quad \frac{((D^{n+2}F(z))')/((D^{n+1}F(z))')+(c-1)}{1+(c-1)((D^{n}F(z))')/((D^{n+1}F(z))')}-(p+1)$$
$$=-p\left[\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-w(z))}{(n+1)(1+w(z))}\right]$$
$$+\frac{2p(1-\alpha)zw'(z)}{(1+w(z))(c(n+1)+(c(n+1)+2p(1-\alpha))w(z))}.$$

The remaining part of the proof is similar to that of Theorem 1.

Putting $p=1$, $a_{-1}=1$, $n=0$ and $\alpha=0$ in the above Theorem 2, we obtain the following result by Goel and Sohi [2].

Corollary. *If*

$$(2.22)\qquad f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k$$

and satisfies the condition

$$(2.23)\qquad \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\frac{1}{2(c+1)}\quad (c>0),$$

then

$$(2.24)\qquad F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt$$

belongs to $\sum_k$.

For $c=1$, the above Corollary extends a result of Bajpai [1].

Theorem 3. *If* $f(z)\in J_n(\alpha)$, *then*

$$(2.25)\qquad F(z)=\frac{1}{z^{1+p}}\int_0^z t^p f(t)dt$$

belongs to $J_n(\alpha)$.

Proof. Since $f(z)\in J_n(\alpha)$ satisfies (2.12), the result follows.

Remark. Taking $p=1$ in above theorems, we have the results by Uralegaddi and Somanatha [6].

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