63. *Gamma Factors and Plancherel Measures*

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We explicitly calculate gamma factors of Selberg zeta functions and give a neat formula to the associated Plancherel measures. This report supplements the previous one [7]. The details are described in [8] and will be published elsewhere.

§ 1. Selberg zeta functions. We fix the notation for Selberg zeta functions following mainly Selberg[13], Gangolli [5], Fried [4] ($\kappa = 1$), and Wakayama [15]. Let $M = \Gamma \backslash G/K$ be a compact locally symmetric space of rank one. We denote by $Z_M(s)$ the Selberg zeta function:

$$Z_M(s) = \prod_{p \in \mathrm{Prim}(M)} \prod_{\lambda \geq 0} (1 - N(p)^{-s-\lambda})$$

where $\mathrm{Prim}(M)$ is the set of prime geodesics of M with the norm function $N(p) = \exp(\mathrm{length}(p))$ and λ runs over a certain semi-lattice. We recall the following fact: $Z_M(s)$ has an analytic continuation to all $s \in \boldsymbol{C}$ as a meromorphic function of order $\dim M$ and has the following functional equation

$$Z_M(2\rho_0 - s) = Z_M(s)\exp\left(\mathrm{vol}(M)\int_0^{s-\rho_0} \mu_M(it)dt\right).$$

Here, $\rho_0 > 0$ and the Plancherel measure $\mu_M(t)$ calculated by Miatello [12] are given as follows (we use renormalized ρ_0, $\mu_M(t)$ and $\mathrm{vol}(M)$ to simplify the constants):

(0) $G = SO(1,2n-1)$ ($\Leftrightarrow \dim M$: odd)
$\rho_0 = n - 1$, $\mu_M(it)$: polynomial

(1) $G = SO(1,2n)$, $\rho_0 = n - 1/2$, $\dim M = 2n$,
$\mu_M(it) = (-1)^n P_M(t)\, \pi \tan(\pi t)$,
$$P_M(t) = \frac{2}{(2n-1)!}\, t \prod_{k=1}^{n-1}\left(t^2 - \left(k - \frac{1}{2}\right)^2\right)$$

(2) $G = SU(1,2n-1)$, $\rho_0 = n - 1/2$, $\dim M = 4n-2$,
$\mu_M(it) = -P_M(t)\, \pi \tan(\pi t)$,
$$P_M(t) = \frac{2}{(2n-1)!(2n-2)!}\, t \prod_{k=1}^{n-1}\left(t^2 - \left(k - \frac{1}{2}\right)^2\right)^2$$

(3) $G = SU(1,2n)$, $\rho_0 = n$, $\dim M = 4n$,
$\mu_M(it) = -P_M(t)\, \pi \cot(\pi t)$,
$$P_M(t) = \frac{2}{(2n)!(2n-1)!}\, t^3 \prod_{k=1}^{n-1} (t^2 - k^2)^2$$

(4) $G = Sp(1, n)$, $\rho_0 = n + 1/2$, $\dim M = 4n$,
$\mu_M(it) = P_M(t)\, \pi \tan(\pi t)$,
$$P_M(t) = \frac{2}{(2n+1)!(2n-1)!}\, t\left(t^2 - \left(n - \frac{1}{2}\right)^2\right) \prod_{k=1}^{n-1}\left(t^2 - \left(k - \frac{1}{2}\right)^2\right)^2$$

(5) $G = \mathrm{F}_4$, $\rho_0 = 11/2$, $\dim M = 16$,
$\mu_M(it) = P_M(t)\, \pi \tan(\pi t)$,

$$P_M(t) = \frac{2}{11!4\cdot 5\cdot 6\cdot 7} t\left(t^2 - \frac{1}{4}\right)^2 \left(t^2 - \frac{9}{4}\right)^2 \left(t^2 - \frac{25}{4}\right) \left(t^2 - \frac{49}{4}\right)\left(t^2 - \frac{81}{4}\right).$$

We omit the case (0) since the gamma factor is "trivial" corresponding to the non-existence of discrete series. In cases (1)–(5) the gamma factor is non-trivial and described by the multiple gamma function of order $\dim M$; we notice that $\deg P_M = \dim M - 1$.

§ 2. Multiple gamma functions and multiple sine functions. We define the multiple gamma function $\Gamma_r(z)$ by

$$\Gamma_r(z) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, z) \Big|_{s=0}\right)$$

where

$$\zeta_r(s, z) = \sum_{n_1,\dots,n_r \geqq 0} (n_1 + \cdots + n_r + z)^{-s} = \sum_{n=0}^{\infty} {}_rH_n\, (n + z)^{-s}$$

is the multiple Hurwitz zeta function. Next we define the multiple sine function $S_r(z)$ by $S_r(z) = \Gamma_r(z)^{-1}\Gamma_r(r - z)^{(-1)^r}$. Among many properties of $S_r(z)$ similar to the usual sine function, the following one is fundamental in this paper.

Theorem 1. *The multiple sine function $S_r(z)$ satisfies the following differential equations*:

(1) $\dfrac{S_r'}{S_r}(z) = (-1)^{r-1}\dbinom{z-1}{r-1}\pi \cot(\pi z)$.

(2) an algebraic differential equation of degree two:

$$S_r''(z) = (1 - P(z)^{-1}) S_r'(z)^2 S_r(z)^{-1} + P'(z)P(z)^{-1}S_r'(z) - \pi^2 P(z) S_r(z)$$

where $P(z) = (-1)^{r-1}\dbinom{z-1}{r-1}$.

When $r = 1$, we see that $S_1(z) = 2\sin(\pi z)$ since $\Gamma_1(z) = (2\pi)^{-1/2}\Gamma(z)$, so (1)(2) are well-known differential equations for the usual sine function. It should be remarked that the multiple gamma function $\Gamma_r(z)$ does not satisfy any algebraic differential equation according to Hölder ($r = 1$) and Barnes [1] ($r \geqq 2$).

§ 3. Gamma factors. Theorem 2. *Let $M = \Gamma\backslash G/K$ be an even dimensional compact locally symmetric space of rank one. Define the gamma factor $\Gamma_M(s)$ of M by*

$$\Gamma_M(s) = \det\left(\sqrt{\Delta_{M'} + \rho_0^2} + s - \rho_0\right)^{\mathrm{vol}(M)(-1)^{\dim M/2}}$$

using the zeta regularized determinant, where $M' = G'/K$ is the compact dual symmetric space. Then:

(1)

$$\Gamma_M(s) = \begin{cases} (\Gamma_{2n}(s)\,\Gamma_{2n}(s+1))^{\mathrm{vol}(M)(-1)^{\dim M/2-1}} \cdots G = \mathrm{SO}(1,2n), \\ (\prod_{k=0}^{n} \Gamma_{2n}(s+k)^{\binom{n}{k}^2})^{\mathrm{vol}(M)(-1)^{\dim M/2-1}} \cdots G = \mathrm{SU}(1,n), \\ (\prod_{k=0}^{2n-1} \Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}})^{-\mathrm{vol}(M)} \cdots G = \mathrm{Sp}(1,n), \\ (\Gamma_{16}(s)\,\Gamma_{16}(s+1)^{10}\,\Gamma_{16}(s+2)^{28}\,\Gamma_{16}(s+3)^{28}\,\Gamma_{16}(s+4)^{10} \\ \qquad \Gamma_{16}(s+5))^{-\mathrm{vol}(M)} \cdots G = \mathrm{F}_4. \end{cases}$$

(2) *The completed zeta funtion* $\hat{Z}_M(s) = \Gamma_M(s) Z_M(s)$ *satisfies the symmetric functional equation*: $\hat{Z}_M(s) = \hat{Z}_M(2\rho_0 - s)$. *Moreover* $\hat{Z}_M(s)$ *is essentially equal to* $\det((\Delta_M - \rho_0^2) + (s - \rho_0)^2)$.

We notice that when M is a compact Riemann surface of genus $g (G = \mathrm{SO}(1,2))$ our normalization of the double gamma function $\Gamma_2(z)$ gives the following neat result:

$$\Gamma_M(s) = \det\left(\sqrt{\Delta_{S^2} + \frac{1}{4}} + s - \frac{1}{2}\right)^{2-2g} = (\Gamma_2(s)\,\Gamma_2(s+1))^{2g-2}$$

and

$$\hat{Z}_M(s) = \det(\Delta_M - s(1-s)) \exp\left((2g-2)\left(s - \frac{1}{2}\right)^2\right).$$

§ 4. **Plancherel measures.** We have the following new expression of the Plancherel measures, which suggests the Betti type interpretation for the coefficients.

Theorem 3.

$$P_M(t + \rho_0) = \begin{cases} {}_{2n}H_t + {}_{2n}H_{t-1} & \cdots G = \mathrm{SO}(1,2n), \\ \sum_{k=0}^{n} \binom{n}{k}^2 {}_{2n}H_{t-k} & \cdots G = \mathrm{SU}(1,n), \\ \sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1} {}_{4n}H_{t-k} & \cdots G = \mathrm{Sp}(1,n), \\ {}_{16}H_t + 10\,{}_{16}H_{t-1} + 28\,{}_{16}H_{t-2} + 28\,{}_{16}H_{t-3} + 10\,{}_{16}H_{t-4} + {}_{16}H_{t-5} & \cdots G = \mathrm{F}_4. \end{cases}$$

§ 5. **Proofs.** We use the following combinatorial result:

Theorem 4. *For integers* n *and* m *we have*:

(1) $${}_{2n}H_m + {}_{2n}H_{m-1} = \frac{(2m+2n-1)(m+1)\cdots(m+2n-2)}{(2n-1)!} = \mathrm{mult}(m(m+2n-1), \Delta_{S^{2n}}).$$

(2) $$\sum_{k=0}^{n} \binom{n}{k}^2 {}_{2n}H_{m-k} = \frac{(2m+n)(m+1)^2\cdots(m+n-1)^2}{n!(n-1)!} = \mathrm{mult}(m(m+n), \Delta_{P^n_{\mathbf{C}}}).$$

(3) $$\sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1} {}_{4n}H_{m-k} = \frac{(2m+2n+1)(m+1)((m+2)\cdots(m+2n-1))^2(m+2n)}{(2n+1)!(2n-1)!} = \mathrm{mult}(m(m+2n+1), \Delta_{P^n_{\mathbf{H}}}).$$

(4) $${}_{16}H_m + 10\,{}_{16}H_{m-1} + 28\,{}_{16}H_{m-2} + 28\,{}_{16}H_{m-3} + 10\,{}_{16}H_{m-4} + {}_{16}H_{m-5} = \frac{(2m+1)(m+1)(m+2)(m+3)(m+4)^2(m+5)^2(m+6)^2(m+7)^2(m+8)(m+9)(m+10)}{11!\,4\cdot5\cdot6\cdot7} = \mathrm{mult}(m(m+11), \Delta_{P^2_{\mathbf{O}}}).$$

The former identities follow from Saalschütz (1890) type identities generalizing Vandermond convolution to three products. The latter identities are due to Cartan [3]((1)(2)) and Cahn-Wolf [2] (general), which are considered as real analytic version of the Hirzebruch proportionality principle since the middle terms are $P_M(m + \rho_0)$. Hence we obtain Theorem 3.

Let

$$\zeta(s, z-\rho_0, \sqrt{\Delta_{M'}+\rho_0^2}) = \textstyle\sum_{m=0}^{\infty} \mathrm{mult}(m(m+2\rho_0), \Delta_{M'})(m+z)^{-s}$$

then Theorem 4 gives (in the same four cases)

$$\zeta(s, z-\rho_0, \sqrt{\Delta_{M'}+\rho_0^2}) = \begin{cases} \zeta_{2n}(s, z) + \zeta_{2n}(s, z+1), \\ \sum_{k=0}^{n} \binom{n}{k}^2 \zeta_{2n}(s, z+k), \\ \sum_{k=0}^{2n-1} \frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1} \zeta_{4n}(s, z+k), \\ \zeta_{16}(s, z) + 10\,\zeta_{16}(s, z+1) + 28\,\zeta_{16}(s, z+2) \\ \quad + 28\,\zeta_{16}(s, z+3) + 10\,\zeta_{16}(s, z+4) \\ \qquad + \zeta_{16}(s, z+5). \end{cases}$$

Thus we get (1) of Theorem 2. Now, Theorem 3 gives

$$\exp\left(\int_0^{s-\rho_0} \mu_M(it)\,dt\right)^{(-1)^{\dim M/2}} = \begin{cases} S_{2n}(s)S_{2n}(s+1), \\ \prod_{k=0}^{n} S_{2n}(s+k)^{\binom{n}{k}^2}, \\ \prod_{k=0}^{2n-1} S_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}}, \\ S_{16}(s)S_{16}(s+1)^{10} S_{16}(s+2)^{28} S_{16}(s+3)^{28} \cdot \\ S_{16}(s+4)^{10} S_{16}(s+5) \end{cases}$$

by logarithmic differentiation using Theorem 1 and remarking that both sides are 1 at $s=\rho_0$. So, we get (2) of Theorem 2.

§ 6. Generalized sine functions. The double sine function was firstly studied by Hölder [6] in 1886. Then, after the almost centennial blank, Shintani [14] in 1977 used it to construct class fields over real quadratic fields. Unfortunately Hölder and Shintani used the notation $F(z)$ and did not name it; the name first appeared in [7]–[9]. We may formulate a version of Kronecker's Jugendtraum as follows: for an integral domain A with the quotient field K, $K^{ab} = K(S_A(K))$ where $S_A(x) = \prod_{a\in A}(a-x)$ is the sine function of A. We refer to [10] (Appendix 1 "A variation of the Kronecker limit formula" 1991 May) concerning established examples of $S_A(x)$ for $A = \mathbf{Z}$ (Kronecker), an imaginary quadratic integer ring (Takagi), and an integer ring of positive characteristic (Carlitz-Drinfeld). General calculations using multiple sine functions $S_r(z\,;(\omega_1,\ldots,\omega_r))$ with parameters satisfying $S_r(z\,;(1,\ldots,1)) = S_r(z)$ were written in [8] containing the partially known real quadratic case due to Shintani [14]. There q-analogues of multiple sine functions were used also.

Multiple sine functions are considered as concrete examples of multiple zeta functions formulated in [9]. We refer to Manin [11] for the excellent exposition from the view point of absolute motives.

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