54. *Asymptotic Expansions of the Mean Square of Dirichlet L-functions*

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1. Introduction. Let χ be a Dirichlet character mod q (q : integer ≥ 2) and let $L(s, \chi)$ with a complex variable $s=\sigma+it$ denote the corresponding Dirichlet L-function. Let h be a fixed non negative integer and $L^{(h)}(s, \chi)$ denote the h-th derivative of $L(s, \chi)$. In this note we consider the asymptotical property of the mean value

(1) $$\varphi(q)^{-1} \sum_{\chi (\mathrm{mod}\, q)} |L^{(h)}(\sigma+it, \chi)|^2,$$

where $\varphi(q)$ is Euler's function and the summation is extended over all the characters mod q.

In the special case $\sigma=\frac{1}{2}$, $t=0$ with $h=0$, by using the Hurwitz zeta-functions, Heath-Brown [1] obtained the asymptotic expression for (1) with respect to the modulus q. In the same direction, Zhang [8], [9] proved, when $h=0, 1$, the similar type of asymptotic formulas for (1) on the critical line $\sigma=\frac{1}{2}$ with $t \geq 3$. (The related articles of Zhang [10]-[12] should be mentioned here.)

On the other hand, Motohashi [6] developed a method to study

$$\varphi(q)^{-1} \sum_{\chi (\mathrm{mod}\, q)} L(u, \chi) L(v, \bar{\chi})$$

as a function of two complex variables u and v, and deduced, when $q=p$ is a prime, the formula

$$(p-1)^{-1} \sum_{\chi (\mathrm{mod}\, p)} \left| L\left(\frac{1}{2}+it, \chi\right)\right|^2 = \log \frac{p}{2\pi} + 2\gamma + \Re \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right) + 2p^{-\frac{1}{2}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \cos(t \log p) - p^{-1} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 + O(p^{-\frac{3}{2}})$$

with Euler's constant γ and the O-constant depending on t, where $\Gamma(s)$, $\zeta(s)$ denote the gamma- and the Riemann zeta-function respectively. An extension of Motohashi's argument yields more precise asymptotic results for (1) with $h=0$ in the region $0<\sigma<1$ and $t \in \boldsymbol{R}$ (cf. [4, Theorems 1 and 2]).

As an application of the saddle point method, we can improve the error estimates of Theorem 2 in [4] as well as Theorem 4 in [2]. In what follows, we give this improvement in a more general form. The detailed proof will appear in [3].

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2. Notations. Let

$$\binom{s}{n} = \frac{\Gamma(s+1)}{n!\Gamma(s-n+1)} \quad (n = 0,1,2,\dots),$$

and we put

$$F(w\,;q) = q^{1-w}\Gamma(w-1)\zeta(w-1), \quad G(u,v) = \frac{\Gamma(1-u)}{\Gamma(v)},$$

$$S_N(u,v\,;k) = \sum_{n=0}^{N-1}\binom{-v}{n}\zeta(u-n)\zeta(v+n)k^{u-n} \quad (N \geq 1),$$

$$P(w\,;q) = \prod_{p|q}(1-p^{-w}),$$

where p runs over all prime divisors of q. By $\mu(n)$ we denote the Möbius function.

3. Results. Then we have the following main theorem:

Theorem. *Let $\boldsymbol{Z}_{\leq 1}$ denote the set of all integers not greater than* 1 *and put* $E = \{\sigma + it\,;\ 2\sigma - 1 \in \boldsymbol{Z}_{\leq 1}$ or $\sigma + it \in \boldsymbol{Z}\}$, *then, for any integer $N \geq 1$, in the region*

$$(2) \qquad \{\sigma + it\,;\ -N+1 < \sigma < N,\ t \in \boldsymbol{R}\}$$

with the exception of the points of E, we have

$$\varphi(q)^{-1}\sum_{\chi(\mathrm{mod}\,q)}|L^{(h)}(\sigma+it,\chi)|^2$$

$$= \frac{d^{2h}}{dw^{2h}}\zeta(w)P(w\,;q)\Big|_{w=2\sigma}$$

$$+ 2P(1\,;q)\sum_{\mu,\nu=0}^{h}\binom{h}{\mu}\binom{h}{\nu}F^{(2h-\mu-\nu)}(2\sigma\,;q)\Re\left\{\frac{\partial^{\mu+\nu}G}{\partial u^\mu\partial v^\nu}(\sigma+it,\sigma-it)\right\}$$

$$+ 2q^{-2\sigma}\sum_{\mu,\nu=0}^{h}\binom{h}{\mu}\binom{h}{\nu}(-\log q)^{2h-\mu-\nu}\sum_{k|q}\mu\left(\frac{q}{k}\right)T^{(\mu,\nu)}(\sigma+it\,;k),$$

where k runs over all positive divisors of q and $T^{(\mu,\nu)}(\sigma+it\,;k)$ has the asymptotic expression

$$T^{(\mu,\nu)}(\sigma+it\,;k) = \Re\left\{\frac{\partial^{\mu+\nu}S_N}{\partial u^\mu\partial v^\nu}(\sigma+it,\sigma-it\,;k) + E_N^{(\mu,\nu)}(\sigma+it\,;k)\right\}.$$

Here $E_N^{(\mu,\nu)}(\sigma+it\,;k)$ is the error term satisfying the estimate

$$(3) \quad E_N^{(\mu,\nu)}(\sigma+it\,;k) = O\left[k^{\sigma-N}(|t|+1)^{2N+\frac{1}{2}-\sigma}\log^{\mu+\nu}\{2k(|t|+1)\}\right]$$

in the region (2), *with the O-constant depending only on σ, N and h. In particular, when $q = p$ is a prime, we have the asymptotic expansion*

$$(p-1)^{-1}\sum_{\chi(\mathrm{mod}\,p)}|L^{(h)}(\sigma+it,\chi)|^2$$

$$= \zeta^{(2h)}(2\sigma) - \frac{\partial^{2h}}{\partial u^h\partial v^h}\{p^{-u-v}\zeta(u)\zeta(v)\}\Big|_{(u,v)=(\sigma+it,\sigma-it)}$$

$$+ 2\sum_{\mu,\nu=0}^{h}\binom{h}{\mu}\binom{h}{\nu}F^{(2h-\mu-\nu)}(2\sigma\,;p)\Re\left\{\frac{\partial^{\mu+\nu}G}{\partial u^\mu\partial v^\nu}(\sigma+it,\sigma-it)\right\}$$

$$+ 2p^{-2\sigma}\sum_{\mu,\nu=0}^{h}\binom{h}{\mu}\binom{h}{\nu}(-\log p)^{2h-\mu-\nu}T^{(\mu,\nu)}(\sigma+it\,;p).$$

From Stirling's formula and the functional equation of $\zeta(s)$, we have

(4)

$$\binom{-\sigma+it}{n}\zeta(\sigma+it-n)\zeta(\sigma-it+n)k^{\sigma+it-n}=O\{k^{\sigma-n}(|t|+1)^{2n+\frac{1}{2}-\sigma}\},$$

for $-n+1<\sigma<n(n\geq 1)$, and the estimate (4) is best-possible because

$$\zeta(\sigma+it)=\Omega(1)$$

for $\sigma>1$ as $|t|\to+\infty$ (cf. Titchmarsh [7, Theorem 8.4]). Hence, when $h=0$, the bound in (3) cannot be replaced by a smaller one.

Moreover, the asymptotic expressions for (1), where $\sigma+it$ lies in the exceptional set E, can be deduced as the limiting cases of our Theorem. For example, we have the following corollary:

Corollary. *Let* $\psi(s)=\frac{\Gamma'}{\Gamma}(s)$ *be the digamma-function and put*

$$A_0(q)=\log\frac{q}{2\pi}+\gamma_0,\quad A_1(q)=\frac{1}{2}\log^2\frac{q}{2\pi}+\gamma_0\log\frac{q}{2\pi}+\gamma_1+\frac{\pi^2}{8},$$

$$A_2(q)=\frac{1}{6}\log^3\frac{q}{2\pi}+\frac{\gamma_0}{2}\log^2\frac{q}{2\pi}+\left(\gamma_1+\frac{\pi^2}{8}\right)\log\frac{q}{2\pi}+\frac{\pi^2}{8}\gamma_0+\gamma_2,$$

where $\gamma_0(=\gamma)$, γ_1, γ_2 *are the coefficients of the Laurent expansion of* $\zeta(s)$ *at* $s=1$ *which are defined by*

$$\zeta(s)=\frac{1}{s-1}+\gamma_0+\gamma_1(s-1)+\gamma_2(s-1)^2+\cdots.$$

Then we have

$$\begin{aligned}
&\varphi(q)^{-1}\sum_{\chi(\mathrm{mod}\,q)}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^2\\
&=P(1;q)\left\{2\gamma_2+2\gamma_1\frac{P'}{P}(1;q)+\gamma_0\frac{P''}{P}(1;q)+\frac{1}{3}\frac{P'''}{P}(1;p)\right.\\
&\quad-\frac{1}{6}\Re\psi''\left(\frac{1}{2}+it\right)+\frac{1}{3}\Re\psi^3\left(\frac{1}{2}+it\right)+A_0(q)\Re\psi^2\left(\frac{1}{2}+it\right)\\
&\quad\left.+2A_1(q)\Re\psi\left(\frac{1}{2}+it\right)+2A_2(q)\right\}\\
&\quad+2q^{-1}\sum_{\mu,\nu=0}^{1}(-\log q)^{2-\mu-\nu}\sum_{k|q}\mu\left(\frac{q}{k}\right)T^{(\mu,\nu)}\left(\frac{1}{2}+it;k\right).
\end{aligned}$$

If $q=p$ *is a prime, then*

$$\begin{aligned}
&(p-1)^{-1}\sum_{\chi(\mathrm{mod}\,p)}\left|L'\left(\frac{1}{2}+it,\chi\right)\right|^2\\
&=2\gamma_2-\frac{1}{6}\Re\psi''\left(\frac{1}{2}+it\right)+\frac{1}{3}\Re\psi^3\left(\frac{1}{2}+it\right)+A_0(p)\Re\psi^2\left(\frac{1}{2}+it\right)\\
&\quad+2A_1(p)\Re\psi\left(\frac{1}{2}+it\right)+2A_2(p)\\
&\quad-\frac{\partial^2}{\partial u\partial v}\{p^{-u-v}\zeta(u)\zeta(v)\}\Big|_{(u,v)=(\frac{1}{2}+it,\frac{1}{2}-it)}\\
&\quad+2p^{-1}\sum_{\mu,\nu=0}^{1}(-\log p)^{2-\mu-\nu}T^{(\mu,\nu)}\left(\frac{1}{2}+it;p\right).
\end{aligned}$$

We note here

$$\frac{P'}{P}(1;q)=\sum_{p|q}\frac{\log p}{p-1},\ \frac{P''}{P}(1;q)=\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2-\sum_{p|q}\frac{p\log^2 p}{(p-1)^2}$$

$$\frac{P'''}{P}(1\,;q)=\Big(\sum_{p|q}\frac{\log p}{p-1}\Big)^3+\sum_{p|q}\frac{p(p+1)\log^3 p}{(p-1)^3}$$
$$-3\Big(\sum_{p|q}\frac{\log p}{p-1}\Big)\Big(\sum_{p|q}\frac{p\log^2 p}{(p-1)^2}\Big).$$

In a similar manner we can deduce the asymptotic formulas for

$$\sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}}|L^{(h)}(1,\chi)|^2,$$

where χ_0 is the principal character mod q (cf. [3] [5]).

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