53. On the π-adic Theory—Galois Cohomology

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In this note, we exhibit, by calculating Galois cohomology, a crucial difference of the π-adic theory in positive characteristic from the usual p-adic theory in characteristic zero. One reason for this difference is that the Carlitz module, which plays in our theory the role of the multiplicative group $\boldsymbol{G}_m$ in the classical theory, is an *additive* group scheme.

Let A be the polynomial ring $\boldsymbol{F}_q[t]$ in one variable t over the finite field $\boldsymbol{F}_q$ of q elements. Let K be a complete discrete valuation field of "mixed characteristic" over A, by which we mean that K is endowed with an injective ring homomorphism $\alpha : A \rightarrow K$ such that the inverse image by α of the maximal ideal of the integer ring of K is a non-zero prime ideal of A. We assume that the residue field of K is perfect. Our objective is to calculate the Galois cohomology group $H^i(\mathrm{Gal}(K^{\mathrm{sep}}/K), \boldsymbol{C}(r))$ for $i = 0$, 1 and $r \in \boldsymbol{Z}$. (The notations are explained below.) Of special importance is that $H^0(\mathrm{Gal}(K^{\mathrm{sep}}/K), \boldsymbol{C}(r))$ does not vanish even if $r \neq 0$. See the concluding Remark 2 for more discussion.

Let π be the unique monic prime element of A such that $\alpha(\pi)$ is a non-unit in the integer ring of K (so (π) is the "residual characteristic" of K). In the following, we think of A as a subring of K by means of α. Let C be the *Carlitz A-module* over A such that the action of $t \in A$ on C is given by $[t](Z) = tZ + Z^q$ with respect to a coordinate Z of C. The π-adic Tate module of C is a rank one free A_π-module, where A_π is the π-adic completion of A. C being considered to be an object over K, the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K acts on $T_\pi(C)$ continuously. (K^{sep} is a fixed separable closure of K. In general, we denote by G_L the absolute Galois group of a field L.) The character $\chi : G_K \rightarrow A_\pi^\times$ which describes this action is called the *Carlitz character.*

For any valuation field L, we denote by $\hat{L}$ the completion of L with respect to the valuation topology. Let $\boldsymbol{C} := \widehat{K^{\mathrm{sep}}}$. The action of G_K on K^{sep} extends uniquely to a continuous action on $\boldsymbol{C}$. $\boldsymbol{C}$ is algebraically closed. For a subfield L of $\boldsymbol{C}$, we denote by L^{rad} the inseparable closure of L in $\boldsymbol{C}$.

For any topological A_π-module M with a continuous G_K-action, and for any $r \in \boldsymbol{Z}$, we define the r-th *Tate twist* $M(r)$ of M by the Carlitz character to be the G_K-module with the same underlying A_π-module M and with a twisted Galois action $\sigma. m = \chi(\sigma)^r \cdot \sigma(m)$ for all $\sigma \in G_K$ and $m \in M$, where $\sigma(m)$ denotes the presupposed action.

For a topological group G and a topological module M with a continuous G-action, we denote by $H^i(G, M)$ the i-th cohomology group defined by the

i-th right derived functor of the functor "fixed part": $M \mapsto M^G$ (or equivalently, defined by continuous cochains). Our main result is:

Theorem. *For all $r \in \boldsymbol{Z}$, we have*

$$(1) \qquad H^0(G_K, \boldsymbol{C}(r)) = \widehat{(K^{\mathrm{rad}} \cdot c^{-r})}(r) \simeq \widehat{K^{\mathrm{rad}}}, \quad \text{and}$$

$$(2) \qquad H^1(G_K, \boldsymbol{C}(r)) = 0.$$

Here c is an element of $\boldsymbol{C}$ such that $\sigma(c) = \chi(\sigma)c$ for all $\sigma \in G_K$, and constructed explicitly in the following.

Remark 1. The followings are previously known:
(i) (Tate [3], Theorems 1 and 2) If K is of characteristic zero and $\boldsymbol{C}_p(r)$ denotes the completion of an algebraic closure of K, with the usual Tate twist, then one has, for $i = 0, 1$,

$$H^i(G_K, \boldsymbol{C}_p(r)) \simeq \begin{cases} K & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases}$$

(ii) (Ax [1]) If K is a rank one valuation field (of arbitrary characteristic) which is henselian with respect to the valuation, then one has

$$H^0(G_K, \boldsymbol{C}) = \widehat{K^{\mathrm{rad}}}.$$

This result includes the case $r = 0$ in (1) of the Theorem.

First of all, note that, when we are working over A_π, we may replace the Carlitz module C by an isomorphic Lubin-Tate A_π-module C' on which the action of π is given by $[\pi](Z') = \pi Z' + Z'^{q^d}$, where $d = \deg(\pi)$. So in the following, we assume $C = C'$, $q = q^d$, and $A_\pi = \boldsymbol{F}_q[[\pi]]$.

We construct now the element $c \in \boldsymbol{C}$. Choose and fix a system $(\pi_n)_{n \geq 0}$ of elements of K^{sep} which corresponds to a generator of $T_\pi(C)$. So π_n is a generator of the π^n-division points of C, and we have $[\pi](\pi_n) = \pi_{n-1}$ for all $n \geq 1$. We define our element $c \in \boldsymbol{C}$ as follows:

$$c := \sum_{n \geq 1} \pi^n \pi_n.$$

The series on the right clearly converges and is non-zero. (1) of the Theorem is implied by Ax's theorem (Remark 1, (ii)) and the following

Lemma 1. *For $x \in \boldsymbol{C}^\times$ and $r \in \boldsymbol{Z}$, write $x = x_1 c^r$ with $x_1 \in \boldsymbol{C}^\times$. Then we have, for all $\tau \in G_K$,*

$$\tau(x) = \tau(x_1)\chi(\tau)^r c^r.$$

In particular, if L is a G_K-stable subfield of $\boldsymbol{C}$ which contains c, then multiplication by c^{-r} induces an isomorphism $L \to L(r)$ of G_K-modules.

Proof. The claim is easily reduced to the case $x = c$ and $r = 1$; we are to show $\tau(c) = \chi(\tau)c$ for all $\tau \in G_K$. Write $f(\pi) = \sum_{i \geq 0} a_i \pi^i$, with $a_i \in \boldsymbol{F}_q$, for the formal power series $\chi(\tau) \in A_\pi^\times$. Then

$$\tau(c) = \sum_{n \geq 1} \pi^n \tau(\pi_n) = \sum_{n \geq 1} \pi^n [f(\pi)](\pi_n) = \sum_{i \geq 0} a_i \sum_{n \geq 1} \pi^n [\pi^i](\pi_n)$$

$$= \sum_{i \geq 0} a_i \pi^i \sum_{n-i \geq 1} \pi^{n-i} \pi_{n-i} = f(\pi)c = \chi(\tau)c.$$

We used in the third equality that the group law of C is $\boldsymbol{F}_q$-linear. Q.E.D.

To prove (2) of the Theorem, we consider certain subextensions of $\boldsymbol{C}/K$ as in [3]. Let K_∞ be the subfield of K^{sep} corresponding to $\mathrm{Ker}(\chi)$; thus the element c is in $\widehat{K_\infty}$, and $\mathrm{Gal}(K_\infty/K)$ is identified with the subgroup $\mathrm{Im}(\chi)$ of

$A_\pi^\times$. Choose

(a) a non-trivial element σ of $\mathrm{Gal}(K_\infty/K)$ such that $\chi(\sigma) \in 1 + \pi A_\pi$, and

(b) a closed subgroup B of $\mathrm{Gal}(K_\infty/K)$

such that $\mathrm{Gal}(K_\infty/K) = \langle\sigma\rangle \times B$, where $\langle\sigma\rangle$ is the closure in $\mathrm{Gal}(K_\infty/K)$ of the cyclic subgroup generated by σ (so $\langle\sigma\rangle \simeq \boldsymbol{Z}_p$, with p the characteristic of K). Denote by L_∞ and M_∞ respectively the subextensions of K_∞ which correspond to B and $\langle\sigma\rangle$. So we have $\mathrm{Gal}(K_\infty/M_\infty) \simeq \mathrm{Gal}(L_\infty/K) \simeq \langle\sigma\rangle$ and $\mathrm{Gal}(K_\infty/L_\infty) \simeq \mathrm{Gal}(M_\infty/K) \simeq B$. The above splitting yields, for each $n \geq 0$, a splitting $\chi^{-1}(1 + \pi^{p^n}A_\pi) = \langle\sigma_n\rangle \times B_n$, where σ_n is a power of σ and B_n is a subgroup of B. Accordingly, we have three fields K_n, L_n and M_n, with $K_n = L_nM_n$, which are the subfields of K_∞ correspoding respectively to $\chi^{-1}(1 + \pi^{p^n}A_\pi)$, $\langle\sigma_n\rangle$ and B_n. Note that $K_n = K(\pi_{p^n})$.

Lemma 2. *Let X be one of the following fields: $\widehat{K_\infty}$, $\widehat{L_\infty}$, $\widehat{K_\infty^{\mathrm{rad}}}$, and $\widehat{L_\infty^{\mathrm{rad}}}$. Then we have $H^1(\langle\sigma\rangle, X) = 0$.*

In fact, as Lemma 3 shows, we have $\widehat{K_\infty^{\mathrm{rad}}} = \widehat{K_\infty}$ and $\widehat{L_\infty^{\mathrm{rad}}} = \widehat{L_\infty}$.

Proof. We prove this for $X = \widehat{K_\infty^{\mathrm{rad}}}$ and $\widehat{L_\infty^{\mathrm{rad}}}$. The other cases are proved in the same way. Since a continuous 1-cocycle: $\langle\sigma\rangle \to X$ is determined by its value at σ, $H^1(\langle\sigma\rangle, X)$ is a subspace of $\mathrm{Coker}(\sigma - 1 : X \to X)$. So it is enough to show the map $\sigma - 1 : X \to X$ is surjective.

For any valuation field F, we denote by $\mathcal{O}_F$ its valuation ring. Let $\mathcal{O}$ be either $\mathcal{O}_{K_\infty^{\mathrm{rad}}}$ or $\mathcal{O}_{L_\infty^{\mathrm{rad}}}$. We first show that $(\sigma - 1)(\mathcal{O})$ contains the maximal ideal of $\mathcal{O}$.

Suppose $\mathcal{O} = \mathcal{O}_{K_\infty^{\mathrm{rad}}}$, and set $\mathcal{O}_n := \mathcal{O}_{K_n^{\mathrm{rad}}}$. For any $n \geq 1$, the map $\sigma_{n-1} - 1 : \mathcal{O}_n \to \mathcal{O}_n$ is $\mathcal{O}_{n-1}$-linear. On the other hand, if n is sufficiently large, there exists an element of $\mathcal{O}_n$ which is mapped by $\sigma_{n-1} - 1$ to an element of $\mathcal{O}_{n-1}$ with absolute value not very small. In fact, if $\chi(\sigma_{n-1}) = 1 + u\pi^k$ with $u \in A_\pi^\times$ and $p^{n-1} \leq k < p^n$, put $m := \min\{p^{n-1} + k, p^n\}$. Then π_m is in $\mathcal{O}_n$, and $(\sigma_{n-1} - 1)(\pi_m) = [u](\pi_{m-k})$ is in $\mathcal{O}_{n-1}$ (Here again we used the additivity of the Carlitz module). Thus $(\sigma_{n-1} - 1)(\mathcal{O}_n)$ contains $\pi_{m-k}\mathcal{O}_{n-1}$. Since σ_{n-1} is a power of σ, $(\sigma - 1)(\mathcal{O}_n)$ also contains $\pi_{m-k}\mathcal{O}_{n-1}$. Passing to the union, and noticing that $m - k$ increases geometrically with n, we see that $(\sigma - 1)(\mathcal{O})$ contains the maximal ideal of $\mathcal{O}$.

The statement for $\mathcal{O} = \mathcal{O}_{L_\infty^{\mathrm{rad}}}$ follows by noting that $\mathcal{O}_{K_\infty^{\mathrm{rad}}}$ is a free $\mathcal{O}_{L_\infty^{\mathrm{rad}}}$-module which admits a free basis consisting of units of $\mathcal{O}_{M_\infty}$. This can be seen, for example, by applying repeatedly the decomposition

$$\mathcal{O}_{L_\infty^{\mathrm{rad}}\cdot M_n} = \bigoplus_{i=0}^{[M_n:M_{n-1}]-1} \mathcal{O}_{L_\infty^{\mathrm{rad}}\cdot M_{n-1}}\cdot \mu_n^i,$$

where μ_n is a unit of $\mathcal{O}_{M_n}$ such that $\mathcal{O}_{M_n} = \mathcal{O}_{M_{n-1}}[\mu_n]$.

Now again let $\mathcal{O}$ be either $\mathcal{O}_{K_\infty^{\mathrm{rad}}}$ or $\mathcal{O}_{L_\infty^{\mathrm{rad}}}$. As above, we can choose a K^{rad}-basis $(\varpi_\nu)_{\nu\geq 0}$ of K_∞^{rad} (resp. L_∞^{rad}) consisting of elements, e.g., of $\pi\mathcal{O}^\times$. Then any element x of X can be written as a convergent series

$$x := \sum_{\nu\geq 0} x_\nu \cdot \varpi_\nu,$$

where $x_\nu \in K^{\mathrm{rad}}$ and $|x_\nu| \to \infty$ as $\nu \to \infty$. Since $\pi\mathcal{O}^\times$ is contained in

$(\sigma-1)(\mathcal{O})$, there exists for each ν an element ϖ'_ν of $\mathcal{O}$ such that $(\sigma - 1)(\varpi'_\nu) = \varpi_\nu$. The element

$$x' = \sum_{\nu \geq 0} x_\nu \cdot \varpi'_\nu \quad \in X$$

is then mapped by $\sigma - 1$ to x. Q.E.D.

The next step is:

Lemma 3 (cf. [3], Proposition 10). *Let K be any complete discrete valuation field with perfect residue field, K_∞ an infinite APF-extension of K ([4]), and L a Galois extension of K_∞. Then we have*

$$H^i(G_{K_\infty}, \hat{L}) = \begin{cases} 0 & \text{if } i > 0, \\ \widehat{K_\infty} & \text{if } i = 0. \end{cases}$$

In particular, we have $\widehat{K_\infty} = \widehat{K_\infty^{\mathrm{rad}}} (= \widehat{K_\infty}^{\mathrm{rad}})$, and hence $\widehat{K_\infty}$ is perfect.

Note that our K_∞, L_∞ and M_∞ are all APF-extensions of K.

As in [3], the above lemma is a formal (though somewhat tricky) consequence of:

Lemma 4 (cf. [3], Proposition 9). *Let K_∞/K be as above, and let L/K_∞ be a finite separable extension. Denote by $\mathcal{O}_L$ the valuation ring of L, and by $\mathfrak{m}_\infty$ the valuation ideal of K_∞. Then we have $\mathrm{Tr}_{L/K_\infty}(\mathcal{O}_L) \supset \mathfrak{m}_\infty$.*

Proof. We reproduce the proof of Tate [3], pointing out how to use our assumption. Replacing K by a finite subextension of L/K, we may suppose that there is a finite extension L_0 of K, linearly disjoint from K_∞, such that $L = L_0 K_\infty$ (see [2], p. 97, Lemma 6). We may also suppose that L_0/K is a Galois extension, because we may replace L/K_∞ by its Galois closure.

For $u \geq -1$, let K_u be the fixed subfield of K_∞ by the u-th ramification group $\mathrm{Gal}(K_\infty/K)^u$ in the upper numbering, and put $L_u := L_0 K_u$. Let v denote the normalized valuation of K. Then the valuation of the different $\mathfrak{D}_{L_u/K_u}$ of L_u/K_u is

$$v(\mathfrak{D}_{L_u/K_u}) = \int_{-1}^{\infty} \left(\frac{1}{(\mathrm{Gal}(K_u/K)^y : 1)} - \frac{1}{(\mathrm{Gal}(L_u/K)^y : 1)} \right) dy.$$

If $h \in \boldsymbol{R}$ is so large that $y \geq h$ implies $\mathrm{Gal}(L/K)^y \subset \mathrm{Gal}(L/L_0)$ (i.e., $\mathrm{Gal}(K_u/K)^y \simeq \mathrm{Gal}(L_u/K)^y$ for all $u \geq -1$), then we have

$$v(\mathfrak{D}_{L_u/K_u}) \leq \int_{-1}^{h} \frac{dy}{(\mathrm{Gal}(K_u/K)^y : 1)}.$$

Since K_∞/K is APF of infinite degree, for any fixed y, $\mathrm{Gal}(K_\infty/K)^y$ is open in $\mathrm{Gal}(K_\infty/K)$ and $(\mathrm{Gal}(K_u/K)^y : 1)$ tends to infinity with u. Hence the above integral tends to zero with u.

Recall (from e.g. [2], p. 60, Proposition 7) that, in general, for a finite integral extension B/A of Dedekind domains and an ideal $\mathfrak{b}$ (resp. $\mathfrak{a}$) of B (resp. A), we have

$$\mathrm{Tr}_{B/A}(\mathfrak{b}) \subset \mathfrak{a} \quad \Leftrightarrow \quad \mathfrak{b} \subset \mathfrak{a}\mathfrak{D}_{B/A}^{-1}.$$

Applying this for $\mathfrak{b} = \mathcal{O}_{L_u}$ and $\mathfrak{a} = \mathrm{Tr}_{L_u/K_u}(\mathcal{O}_{L_u})$, we see that

$$\mathfrak{D}_{L_u/K_u} \subset \mathrm{Tr}_{L_u/K_u}(\mathcal{O}_{L_u})\mathcal{O}_{L_u}.$$

Since $v(\mathfrak{D}_{L_u/K_u}) \to 0$ as $u \to \infty$, so does $v(\mathrm{Tr}_{L_u/K_u}(\mathcal{O}_{L_u})\mathcal{O}_{L_u})$. This means that $\mathrm{Tr}_{L/K_\infty}(\mathcal{O}_L) \supset \mathfrak{m}_\infty$. Q.E.D.

Now we can complete the proof of (2) of the Theorem. By Lemma 1, we may assume $r=0$. Look at the spectral sequence

$$0 \to H^1(\mathrm{Gal}(L_\infty/K),\ H^0(G_{L_\infty},\ \boldsymbol{C})) \to H^1(G_K,\ \boldsymbol{C}) \to H^1(G_{L_\infty},\ \boldsymbol{C}).$$

By Lemma 3, $H^1(G_{L_\infty}, \boldsymbol{C}) = 0$. By Ax (Remark 1, (ii)), $H^0(G_{L_\infty}, \boldsymbol{C}) = \widehat{L_\infty^{\mathrm{rad}}}$. By Lemma 2, $H^1(\mathrm{Gal}(L_\infty/K), \widehat{L_\infty^{\mathrm{rad}}}) = 0$. Hence we obtain (2).

Remark 2. Lemma 1 shows that $\boldsymbol{C}$ is (and in fact, even $\widehat{K_\infty}$ is) "so big" that a topological $A_\pi[G_K]$-module loses much information after being tensored with $\boldsymbol{C}$. This is because we have our element c in $\boldsymbol{C}$, and at this point, our $\boldsymbol{C}$ might be more analogous to $\boldsymbol{B}_{\mathrm{dR}}$ or $\boldsymbol{B}_{\mathrm{cris}}$ in the usual p-adic theory, rather than to $\boldsymbol{C}_p = \widehat{\boldsymbol{Q}_p^{\mathrm{sep}}}$ (this observation was communicated to the author by Nobuo Tsuzuki, to whom the author is grateful). But our $\boldsymbol{C}$ does not have enough structures to recover π-adic Galois representations. Is there a cleverer ring than $\boldsymbol{C}$?

References

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