6. *The Determination of the Imaginary Abelian Number Fields with Class Number One*

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Uchida proved that there exist only finitely many imaginary abelian number fields with class number one [12], and gave the value 2×10^{10} as an upper bound of the conductors of such fields [13]. Several authors determined such fields of some types, but not all of them have been determined yet. (Masley determined the cyclotomic number fields with class number one [8], and Uchida determined such fields of two power degrees [14].)[1] The purpose of this article is to report that we have determined all the imaginary abelian number fields with class number one in proving the following. (The details will appear elsewhere [15].)

Theorem. *There exist exactly* 171 *imaginary abelian number fields with class number one as given in the attached table. Among them,* 29 *fields are cyclotomic,* 49 *fields are cyclic, and* 87 *fields are maximal with respect to inclusion. The maximal conductor of these fields is* $10921=67\cdot 163$, *which is the conductor of the biquadratic number field* $\boldsymbol{Q}(\sqrt{-67},\sqrt{-163})$.

Now we sketch here the method of proof. The basic idea is due to Uchida. (See [13] [14].) In the following, let K be an imaginary abelian number fields with class number $h(K)$. When $h(K)=1$, the genus number[2] of K is one, which is equivalent to say that the character group X corresponding to K is a direct product of subgroups generated by a character of prime power conductor:

$$X=\langle\chi_1\rangle\times\cdots\times\langle\chi_r\rangle,\ \text{the conductor } f_{\chi_i} \text{ of } \chi_i=\text{a prime power.}$$

Moreover, when $h(K)=1$, the subfield of K corresponding to $\langle\chi_i\rangle$ has strict class number one ([13], Prop. 1). Therefore, we need to consider only K of this type. Among K of this type, we first determine K with $h^-(K)=1$, and then check whether $h^+(K)=1$, or not. Here, $h^-(K)$ (resp. $h^+(K)$) denotes the relative class number of K (resp. the class number of the maximal real subfield of K). ($h(K)=h^-(K)h^+(K)$.) The key idea which facilitates the determination is:

(∗) If $h(K)=1$, then, for any subfield F of K such that K/F is totally ramified at a finite prime, the strict class number of F is one.

From this, in most cases, we can immediately restrict K to be considered

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[1] Recently, Louboutin determined imaginary abelian sextic number fields with class number one [4]. The author knew his result after completion of the present article.

[2] The extension degree $[K^*, K]$, where K^* is the genus field of K.

by using information on fields with smaller character group. Thus it is reasonable to start with small character group. Since K is imaginary, at least one χ_t is odd. We may and shall always take χ_1 as an odd generator of minimal order. We first treat the case where the conductor of K is a power of a prime number $p(>2)$. In this case, if $h(K)=1$, then the subfield of K corresponding to the 2-Sylow subgroup of X is also an imaginary abelian number field with class number one, which is known by Uchida [14]. Therefore, the possibilities of the values of p are known and their determination is easy. Next, we treat the case where $r=2$ and $\operatorname{ord}\chi_1=$a power of two, $\operatorname{ord}\chi_2=$a prime (>2). In this case, we need the help of computer for the calculation of relative class numbers of many fields. The other cases are easily treated by using $(*)$.

In order to decrease the amount of computer calculation, we need a good lower bound of $h^-(K)$. This problem is reduced to that of getting a good upper bound of $|L(1,\chi)|$ for nontrivial even Dirichlet characters $\chi \in X$ by using Uchida's estimation for $h^-(K)$ ([14], Prop. 1) and several results on real zeros of the Dedekind zeta function (Heilbronn [2], Low [5], and Rosser [11]). Slightly extending Moser's result ([9], §3), we get

Proposition. *Let χ be a primitive nontrivial even Dirichlet character of conductor f. If the number of prime divisors of f is less than or equal to three, then*

$$|L(1,\chi)|<\frac{1}{2}\log f+\gamma-\frac{1}{2},$$

Table of the imaginary abelian number fields with class number one

t	r	Type	Degree	Fields
1	1	2*	2	3; 4; 7; 8; 11; 19; 43; 67; 163
		4*	4	5; 13; 16; 29; 37; 53; 61
		6*	6	7; 9; 19; 43; 67
		8*	8	32; 41
		10*	10	11
		12*	12	13; 37; 61
		14*	14	43; 49
		16*	16	17
		18*	18	19; 27
		20*	20	25
	2	(2*, 2)	4	(4, 8)
		(2*, 4)	8	(4, 16)
		(2*, 8)	16	(4, 32)

(continued)

t	r	Type	Degree	Fields
2	2	(2*,2)	4	(3,5); (3,8); (3,17); (3,41); (3,89); (4,5); (4,13); (4,37); (7,5); (7,13); (7,61); (8,5); (8,29); (11,8); (11,17)
		(2*,2*)	4	(3,4); (3,7); (3,8); (3,11); (3,19); (3,43); (3,67); (3,163); (4,7); (4,11); (4,19); (4,43); (4,67); (4,163); (7,8); (7,11); (7,19); (7,43); (7,163); (8,11); (8,19); (8,43); (8,67); (11,19); (11,67); (11,163); (19,67); (19,163); (43,67); (43,163); (67,163)
		(2*,3)	6	(3,7); (3,13); (3,31); (3,43); (4,7); (4,9); (4,19); (7,9); (7,13); (8,7); (8,13); (11,7)
		(2*,4)	8	(3,16)
		(2*,4*)	8	(3,5); (3,16); (4,5); (4,13); (4,37); (7,5); (7,13); (8,5); (8,29); (11,16)
		(2*,5)	10	(3,11); (4,11)
		(2*,6*)	12	(3,7); (3,43); (4,7); (4,9); (4,19); (7,9); (8,7); (11,7)
		(2*,8*)	16	(3,32)
		(2*,10*)	20	(3,11); (4,11)
		(4*,2)	8	(5,8); (5,13); (5,17); (13,5); (13,8); (16,5)
		(4*,3)	12	(5,7); (5,9)
		(4*,4*)	16	(5,13); (5,16)
		(4*,6*)	24	(5,7); (5,9)
		(6*,2)	12	(7,5); (9,5); (9,8)
		(6*,3)	18	(9,7)
	3	(2*,2,2)	8	(4,8,5)
		(2*,2,2*)	8	(4,8,3); (4,8,11)
		(2*,2,3)	12	(4,8,7)
		(2*,2,4*)	16	(4,8,5)
		(2*,4,2)	16	(4,16,5)
		(2*,4,2*)	16	(4,16,3)
3	3	(2*,2,2*)	8	(3,5,4); (3,5,7); (3,5,8); (3,8,11); (3,17,11); (4,5,7); (4,13,7)
		(2*,2,3)	12	(3,5,7)
		(2*,2*,2*)	8	(3,4,7); (3,4,11); (3,4,19); (3,7,8); (3,11,19); (4,7,19)
		(2*,2*,3)	12	(3,4,7); (3,7,13); (3,8,7); (3,8,13), (3,11,7); (4,7,9); (4,11,7); (7,8,13)
		(2*,2*,4*)	16	(3,4,5); (3,7,5); (3,8,5); (4,7,5)
		(2*,2,5)	20	(3,4,11)
		(2*,2*,6*)	24	(3,4,7)
		(2*,3,4*)	24	(3,7,5)

where γ is the Euler constant.

This enables us to execute all the calculation on personal computer NEC PC-9801. The restriction on f in Proposition can be probably removed, but the author has not succeeded yet.

Most part of the check $h^+(K)=1$ is done by using results of Masley [7] and Mäki [6].

In the table, t in the first column means the number of prime divisors of the conductor of a field and $*$ means the oddness (imaginarity) of a generator of character group. Each field K with $h(K)=1$ is expressed as $(f_{\chi_1}, \cdots, f_{\chi_r})$ for each type $(\operatorname{ord}\chi_1, \cdots, \operatorname{ord}\chi_r)$ of X. For example, for the field expressed as $(3, 4, 7)$ with type $(2^*, 2^*, 6^*)$, the character group is generated by three odd characters of order 2, 2, 6 and of conductor 3, 4, 7, respectively. Hence, this field is the cyclotomic number field $\boldsymbol{Q}(\zeta_3, \sqrt{-1}, \zeta_7) = \boldsymbol{Q}(\zeta_{84})$.

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