# 51. *Description of Sequences Defined by Billiards in the Cube*

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§ 1. **Introduction.** We consider billiards in the cube $I^3$, $I=[0, 1]$, whose faces $\{\delta\}\times I\times I$, $I\times\{\delta\}\times I$, and $I\times I\times\{\delta\}$ are labelled by $a$, $b$, and $c$, resp., where $\delta=0$ or $1$ and $A\times B\times C=\{(x, y, z)\mid x\in A, y\in B, z\in C\}$. Let a particle start at a point $P\in F$ with constant velocity along a vector $v=(1, \alpha, \beta)$ and reflected at each face specularly, where $F=\{0\}\times I'\times I'\cup I'\times\{0\}\times I'\cup I'\times I'\times\{0\}$, $I'=[0, 1)$. We assume that
(A) $\alpha, \beta, \beta/\alpha$ are irrational with $1>\alpha>\beta>0$, and
(B) the (forward) path of the particle never touch the edges of the cube.
$A$ point $P\in F$ of the property (B) will be called *lattice-free* w.r.t, a given $v$. If we write down the labels $a$, $b$, and $c$ of the faces which the particle hits in order of collision, we have an infinite sequence, or word,

$$w=w(v, P)=w(v, P; a, b, c)\in\{a, b, c\}^{N}.$$

In [1] the authors jointly with P. Arnoux and C. Mauduit proved the following theorem conjectured by G. Ranzy: If $1, \alpha, \beta$ are linearly independent over $Q$ and if $P\in F$ is lattice-free w. r. t. $v$, then the *complexity* $p(n; w)$ of the word $w=w(v, P)$ is given by

$$p(n; w)=n^2+n+1 \quad (n\geq 1),$$

where $p(n; w)$ is, by definition, the number of distinct subwords of $w$ of length $n$. The purpose of this note is to give an algorithm describing the word $w$ in terms of the *partial quotients* of the simple continued fractions of $\alpha$, $\beta$, and $\beta/\alpha$ and the *digits* appearing in certain expansions, defined by (4) below, of the coordinates of the point $P$.

By symmetry with respect to the faces, the word $w$ remains unchanged, if we replace the cube by the three dimensional torus $\boldsymbol{R}^3/\boldsymbol{Z}^3$ and imagine that the particle does not reflect at the faces but passes through them. If we attach the symbols $a$, $b$, and $c$ to the intersection points of the half-line $l=\{tv+P\mid t>0\}$ to the planes $x=k\in\boldsymbol{N}$, $y=m\in\boldsymbol{N}$, and $z=n\in\boldsymbol{N}$, resp., and trace them along $l$, we obtain the word $w(v, P)$ defined above. We remark that a point $P=(\xi, \eta, \zeta)\in F$ is lattice-free w.r.t. $v=(1, \alpha, \beta)$ if and only if

(1) $$k\theta_i+\phi_i\notin\boldsymbol{Z} \text{ for all } k\in\boldsymbol{N}\ (i=1, 2, 3),$$

where

(2) $$\theta_1=\alpha,\ \phi_1=\eta-\alpha\xi,\ \theta_2=\beta,\ \phi_2=\xi-\beta\xi,\ \theta_3=\frac{\beta}{\alpha},\ \phi_3=\xi-\frac{\beta}{\alpha}\eta,$$

and that almost all points $P\in F$ in the sense of Lebesgue Measure are lattice-free w.r.t. a given vector $v$.

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**§ 2. The sequence** $\{[(k+1)\theta+\phi]-[k\theta+\phi]\}_{k\geq 1}$. One of the algorithm writing down the sequence in the title of this section was given by the authors and K. Nishioka [2], which will be used to describe the word $w=w(v, P)$ defined in the previous section. Here $[x]$ denotes the greatest integer not exceeding a real number $x$. We remark that if $\phi=0$ the result (see Theorem A. (i) below) is classical (cf. [3]).

Let $\theta=[a_0; a_1, a_2, \cdots]$ denote the simple continued fraction of $\theta$, where $\theta=a_0+\theta_0$, $a_0=[\theta]$, and $1/\theta_{n-1}=a_n+\theta_n$, $a_n=[1/\theta_{n-1}]$ $(n\geq 1)$. We expand $\phi$ in terms of the sequence $\{\theta_n\}_{n\geq 0}$. Put $\phi=b_0-\phi_0$, $b_0=-[-\phi]$, and define

(3) $$\phi_{n-1}/\theta_{n-1}=b_n-\phi_n,\ b_n=-[-\phi_{n-1}/\theta_{n-1}]\ (n\geq 1).$$

Then $\phi$ can be expanded in the series

(4) $$\phi=b_0+\sum_{n=1}^{\infty}(-1)^n\theta_0\theta_1\cdots\theta_{n-1}b_n=b_0.b_1b_2\cdots.$$

By definition, $0\leq\phi_n<1$ $(n\geq 0)$ and $b_n\in\boldsymbol{Z}$ with $0\leq b_n\leq a_n+1$ $(n\geq 1)$. The series terminates if and only if $b_{n+1}=0$ for some $n\geq 0$; and if so $b_n=0$ for all $n\geq N=\min\{\nu\geq 1 \mid b_\nu=0\}$. Otherwise, $b_n\geq 1$ for all $n\geq 1$.

**Theorem A** ([2]). *Let $\theta$ be irrational with $0<\theta<1$ and $\phi$ be real.*

(i) *If $\phi$ is an integer, we have*

$$\{[(k+1)\theta+\phi]-[k\theta+\phi]\}_{k\geq 1}=\{[(k+1)\theta]-[k\theta]\}_{k\geq 1}=\lim_{n\to\infty} w_n,$$

*where $w_n$ is given by*

$$w_0=0,\ w_1=\underbrace{0\cdots 0}_{a_1-1\ times}1,\ w_n=\underbrace{w_{n-1}\cdots w_{n-1}}_{a_n\ times}w_{n-2}\quad (n\geq 2).$$

(ii) *If $\phi$ is not an integer and if*

$$k\theta+\phi\notin\boldsymbol{Z}\text{ for all }k\in\boldsymbol{N},$$

*we have*

$$\{[(k+1)\theta+\phi]-[k\theta+\phi]\}_{k\geq 1}=\lim_{n\to\infty} u_n v_n,$$

*where $u_n$ and $v_n$ are defined by*

$$v_0=0,\ u_1=\underbrace{0\cdots 0}_{b_1\ times},\ v_1=\underbrace{0\cdots 0}_{a_1-1\ times}1,$$

$$u_n=u_{n-1}\underbrace{v_{n-1}\cdots v_{n-1}}_{b_n\ times},\ v_n=\underbrace{v_{n-1}\cdots v_{n-1}}_{a_n\ times}v_{n-2}\quad (n\geq 2),$$

*and $\lim_{n\to\infty} x_n$ denotes the infinite word having $x_n$ as a prefix for all $n$.*

**Remark.** If $k_0\theta+\phi=m_0$ for some integers $k_0\geq 1$ and $m_0$, we have $[(k+1)\theta+\phi]-[k\theta+\phi]=[(k-k_0+1)\theta]-[(k-k_0)\theta]$, so that this case can be reduced to Case (i).

Theorem A has a natural interpretation into the language of substitutions.

**Theorem A′** ([2]). *Let $\theta$ be irrational with $0<\theta<1$ and $\phi$ be real.*

(i) *We have*

$$\{[(k+1)\theta]-[k\theta]\}_{k\geq 1}=\lim_{n\to\infty}\sigma_{a_1-1}\circ\sigma_{a_2}\circ\cdots\circ\sigma_{a_n}(0),$$

*where* $\sigma_i$ *is the substitution over* $\{0, 1\}$ *defined by*

$$\sigma_i(0) = \underbrace{0 \cdots 0}_{i \text{ times}} 1,\ \sigma_i(1) = 0,$$

*and the product* $\sigma \circ \tau$ *of two substitution* $\sigma$ *and* $\tau$ *is defined by* $\sigma \circ \tau(u) = \sigma(\tau(u))$, $u \in \{0, 1\}.^*$ ($\{a, b, \ldots, d\}^*$ *denotes the set of all finite words on* $a, b, \ldots, d$ *including the empty word.*)

(ii) *If* $\phi \notin \boldsymbol{Z}$ *satisfies* (5), *we have*

$$\{[(k+1)\theta + \phi] - [k\theta + \phi]\}_{k \geq 1} = \lim_{n \to \infty} \sigma_{a_1-1\, b_1} \circ \sigma_{a_2 b_2} \circ \cdots \circ \sigma_{a_n b_n}(\varepsilon),$$

*where the left-hand side is the infinite word of* 0 *and* 1 *prefixed by an auxiliary symbol* $\varepsilon$, *and* $\sigma_{ij}$ *is the substitution over* $\{\varepsilon, 0, 1\}$ *defined by* $\sigma_{ij}(\varepsilon) = \varepsilon\underbrace{0 \cdots 0}_{j \text{ times}}$, $\sigma_{ij}(0) = \underbrace{0 \cdots 0}_{i \text{ times}} 1$, $\sigma(1) = 0$.

**§ 3. Words defined by billiards on the square.** In this section, we consider billiards on the square $I^2$ whose sides $\{\delta\} \times I$ and $I \times \{\delta\}$ are labelled by $a$ and $b$, resp., where $\delta = 0$ or 1. Let a particle start at a point $P \in [0, 1)^2$ with constant velocity along a vector $v = (1, \theta)$ and reflected at each side specularly. We assume that

(A′) $\theta$ is irrational with $0 < \theta < 1$, and

(B′) the (forward) path of the particle never touch the corners of the square.

A point $P \in [0, 1)^2$ of the property (B′) will be called *lattice-free* w.r.t. $\theta$. Writing down the labels $a$ and $b$ of the sides which the particle hits in order of collision, we have an infinite word

(6) $$w = w(\theta, P) = w(\theta, P\,;\, a, b).$$

The word $w(\theta, P)$ remains unchanged, if we replace the sequence by the torus and imagine that the particle does not reflect at the sides but passes through them. If we attach the symbols $a$ and $b$ to the intersection points of the half-line $y = \theta x + \phi$ $(x > 0,\ \theta = \eta - \theta\xi)$ with the lines $x = k \in \boldsymbol{N}$ and $y = m \in \boldsymbol{N}$, resp., and trace them along the half-line, we obtain the word $w(\theta, P)$ defined above. We remark that, if $P = (\xi, \eta) \in [0, 1)^2$, $-\theta < \phi = \eta - \theta\xi < 1$. Thus we have the following

**Lemma 1.** *Let* $P \in [0, 1)^2$ *be lattice-free w.r.t. a given irrational* $\theta$ *with* $0 < \theta < 1$. *Then we have*

$$w(\theta, P) = m_0\, m_1\, m_2 \cdots,$$

*where*

$$m_0 = \begin{cases} \lambda & \text{if } -\theta < \phi < 1 - \theta, \\ b & \text{if } 1 - \theta < \phi < 1, \end{cases}$$

$$m_k = \begin{cases} a & \text{if } [(k+1)\theta + \phi] - [k\theta + \phi] = 0, \\ ab & \text{if } [(k+1)\theta + \phi] - [k\theta + \phi] = 1, \end{cases}$$

*and* $\phi = \eta - \theta\xi$, *wher* $\lambda$ *is the empty word.*

We remark that a point $P \in [0, 1)^2$ is lattice-free w.r.t. $v = (1, \theta)$ if and only if (5) holds with $\phi = \eta - \theta\xi$, and that all the points $P \in \{0\} \times [0, 1) \cup [0, 1) \times \{0\}$, except countable many ones, are lattice-free w.r.t. a given irrational $\theta$ with $0 < \theta < 1$.

**§ 4. An algorithm describing the words $w(v, P)$.** For any $x \in \{a, b, c\}$ and any word $w$ over $\{a, b, c\}$, let $\tau_x(w)$ denote the word obtained by removing $x$ from $w$. We note that $\tau_x(uv) = \tau_x(u)\,\tau_x(v)$ for any $u, v \in \{a, b, c\}^*$. Then by projecting the billiards in $\boldsymbol{R}^3$ along $z$, $y$, and $x$-axis, we have the following

**Lemma 2.** *Let $P = (\xi, \eta, \zeta) \in [0, 1)^3$ be lattice-free w.r.t. a given $v = (1, \alpha, \beta)$ satisfying* (A) *and let $w = w(v, P\,; a, b, c)$ be the word defined in section one. Then we have*

$$\tau_c(w) = w(\alpha, (\xi, \eta)\,; a, b),$$
$$\tau_b(w) = w(\beta, (\xi, \zeta)\,; a, c),$$
$$\tau_a(w) = w(\beta/\alpha, (\eta, \zeta)\,; b, c),$$

*where for instance the right-hand side of the last equality is the word defined by* (6) *with $\theta = \beta/\alpha$, $P = (\eta, \zeta)$, and $b$, $c$ in place of $a$, $b$.*

For any word $m = m_1 m_2 \cdots m_\ell$ with $m_i \in \{a, b, c\}$ $(1 \leq i \leq \ell)$, let $|m|$ denote the length $\ell$ of $m$; in particular, $|\lambda| = 0$. Now we state our theorem.

**Theorem.** *Let $w = w(v, P)$ be the word defined by a lattice-free point $P \in [0, 1)^3$ w.r.t. a given $v$ satisfying* (A). *Then $w$ can be written by the following algorithm:*

*Step* 1. *Expand $\theta_i$ and $\phi_i$ $(i = 1, 2, 3)$ defined by* (2) *in the simple continued fractions*

$$\theta_i = [0\,; a_1^{(i)}, a_2^{(i)}, \cdots]$$

*and then in the series* (4)

$$\phi_i = b_0^{(i)},\ b_1^{(i)}\, b_2^{(i)} \cdots.$$

*Step* 2. *Write down the sequences*

(7) $$\{[(k+1)\theta_i + \phi_i] - [k\theta_i + \phi_i]\}_{k \geq 1} \quad (i = 1, 2, 3)$$

*as the word of* 0 *and* 1, *using Theorem* A.

*Step* 3. *Write the words $\tau_c(w)$, $\tau_b(w)$, $\tau_a(w)$ by Lemma* 1 *with* (7) *as*

$$\tau_c(w) = s_0 s_1 s_2 \cdots,\ s_0 \in \{\lambda, b\},\ s_n \in \{a, ab\}\ (n \geq 1),$$
$$\tau_b(w) = t_0 t_1 t_2 \cdots,\ t_0 \in \{\lambda, c\},\ t_n \in \{a, ac\}\ (n \geq 1),$$
$$\tau_a(w) = u_0 u_1 u_2 \cdots,\ u_0 \in \{\lambda, c\},\ u_n \in \{b, bc\}\ (n \geq 1).$$

*Step* 4. *Rewrite $\tau_a(w)$ in the form*

$$\tau_a(w) = v_0 v_1 v_2 \cdots,\ v_n \in \{\lambda, b, c, bc, cb\},\ (n \geq 1),$$

*where*

$$|v_0| = |s_0| + |t_0|,\ |v_n| = |s_n| + |t_n| - 2\ (n \geq 1).$$

*Step* 5. *Put $w_0 = v_0$, $w_n = av_n$ $(n \geq 1)$. Then the word $w$ is given by*

$$w = w_0 w_1 w_2 \cdots.$$

*Proof of Theorem.* We have only to verify the last step. By Lemma 2, we see that $s_n = \tau_c(w_n)$, $t_n = \tau_b(w_n)$, and $v_n = \tau_a(w_n)$ $(n \geq 0)$. Therefore $w_n$ $(n \geq 1)$ are determined as in the following;

| $a_n$ | $t_n$ | $v_n$ | $w_n$ |
|---|---|---|---|
| $a$ | $a$ | $\lambda$ | $a$ |
| $ab$ | $a$ | $b$ | $ab$ |
| $a$ | $ac$ | $c$ | $ac$ |
| $ab$ | $ac$ | $bc$ | $abc$ |
| $ab$ | $ac$ | $cb$ | $acb$ |

## References

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[3] K. B. Stolarsky: Beatty sequences, continued fractions, and certain shift operators. Canad. Math. Bull., **19**, 473–482 (1976).