47. *Extension of Jones' Projections*

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Introduction. In the index theory for a pair of type II_1-factors, Jones' projections play an important role. A family of Jones' projections is a sequence of projections $\{e_i\,;\, i=1,2,\cdots\}$ satisfying the following condition which we call Jones' relations:

(a) $e_ie_{i\pm1}e_i=\lambda e_i$ for $i\geq 1$ with a fixed constant λ $(0<\lambda<1)$,

(b) $e_ie_j=e_je_i$ for $|i-j|\geq 2$,

(c) $tr(e_i\omega)=\lambda tr(\omega)$ for any word ω on $e_1,\ldots,e_{i-1}$,

where tr is the canonical trace on $\{e_i\,;\, i=1,2,\cdots\}''$.

In this paper, we extend such a family by adding some number of projections. A neccesary and sufficient condition for the existence of such a family is given by Theorems 1 and 2. For a family of extended Jones' projections $\{e_i, f_j\,;\, i=1,2,\cdots, 1\leq j\leq m\}$, put $A=\{e_i, f_j\,;\, i=1,2,\cdots, 1\leq j\leq m\}''$ and $B=\{e_i\,;\, i=1,2,\cdots\}''$. We calculate the index $[A:B]$ and show that the relative commutant $B'\cap A$ is trivial. Furthermore we specify the fixed point subalgebras $A^{\sigma}\subset A$ of automorphisms $\sigma: A\to A$, defined by permutations of $\{f_i\,;\, 1\leq i\leq m\}$, and then calculate indices $[A:A^{\sigma}]$.

§1. Family of extended Jones' projections.

Definition 1. Let $m, n\in \boldsymbol{N}$ and $\{e_i, f_j\,;\, i\geq 1, 1\leq j\leq m\}$ be a family of non-zero projections of M, *a type II_1-factor,* such that

(R-1) $e_ie_{i+1}e_i=\lambda e_i$ for $i\geq 1$,

(R-2) $e_ie_{i-1}e_i=\lambda e_i$ for $i\geq 2$; $e_1f_je_1=\alpha_je_1$ for $1\leq j\leq l$,

(R-3) $e_ie_j=e_je_i$ for $|i-j|\geq 2$; $e_if_j=f_je_i$ for $i\geq 2$, $1\leq j\leq l$,

(R-4) $tr(e_i\omega)=\lambda tr(\omega)$ for any word ω on $1, f_1,\ldots, f_m, e_1,\ldots, e_{i-1}$,

where tr is the canonical trace on M,

(R-5) $\sum_j f_j=1$,

where $\lambda^{-1}=4\cos^2(\pi/(n+2))$, $\alpha_j\in\boldsymbol{R}$, $0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_m$. We call the above relations (R-1)~(R-5) the *extended Jones' relations*, and projections $\{e_i, f_j\,;\, i\geq 1, 1\leq j\leq m\}$ *extended Jones' projections.*

Theorem 1. *Let M be a type II_1-factor. If there exists a family of extended Jones' projections corresponding to the data $(n\,;\,\alpha_1,\ldots,\alpha_m)$, then $(n\,;\,\alpha_1,\ldots,\alpha_m)$ is one of the following:*

$$(n\,;\,\lambda_k,\lambda_{n-k-2})\ \left(0\leq k\leq\left[\frac{n-2}{2}\right]\right),\ (2k\,;\,\lambda_0,\lambda_0,\lambda_{k-2})\ (k\geq 2),$$

$$(10\,;\,\lambda_0,\lambda_1,\lambda_1),\ (16\,;\,\lambda_0,\lambda_1,\lambda_2),\ (28\,;\,\lambda_0,\lambda_1,\lambda_3),$$

where $\lambda_k=\sin(k+1)\theta_n/(2\cos\theta_n\sin(k+2)\theta_n)$ *and* $\theta_n=\pi/(n+2)$.

Proof. Since a sequence $\{f_i, e_1, e_2, \cdots\}$ is a tower of projections corresponding to $\{\alpha_j, \lambda, \lambda, \cdots\}$, α_j must be one of $\{\lambda_j ; 0 \leq j \leq n-2\}$ in Cor. 2. 11 of [4], and $\lambda = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$. So we have $\alpha_j \geq \lambda$. Moreover from relation (R-5), we get $1 = \sum_{j=1}^{m} \alpha_j \geq m\lambda$, hence $m \leq \lambda^{-1}$. Hence $m = 2$ or 3 and $\lambda^{-1} \geq m$.

(a) CASE OF $m = 2$: By simple calculation, we get $\lambda_k + \lambda_{n-k-2} = 1$. So we obtain $(n ; \alpha_1, \alpha_2) = (n ; \lambda_k, \lambda_{n-k-2})$ for some k, $0 \leq k \leq \left[\frac{n-2}{2}\right]$.

(b) CASE OF $m = 3$: Since $\lambda^{-1} \geq m$, $\lambda^{-1} = 4\cos^2(\pi/(n+2))$, we have $n \geq 4$. And $\alpha_1 \leq \alpha_j$ implies that $\alpha_1 \leq 1/3$. On the other hand $1/3 < \lambda_1 < \cdots < \lambda_{n-1}$, so $\alpha_1 = \lambda_0$. By $\lambda_0 + \alpha_1 + \alpha_2 = 1$ and $\alpha_2 \leq \alpha_3$, we get $\alpha_2 \leq (1-\lambda_0)/2$. Moreover $\lambda_2 > (1-\lambda_0)/2$ and so $\alpha_2 = \lambda_0$ or λ_1.

b$_1$) CASE OF $\alpha_2 = \lambda_0$: Since $\alpha_3 = 1 - 2\lambda_0 \in \{\lambda_i ; 0 \leq i \leq n-1\}$, we have $\alpha_3 = \lambda_k$ for some k, $0 \leq k \leq n-1$. Then $\lambda_k = 1 - 2\lambda_0$ By a simple calculation, $n = 2k + 4$.

b$_2$) CASE OF $\alpha_2 = \lambda_1$: Here $\alpha_3 = 1 - \lambda_0 - \lambda_1$. We obtain $\alpha_3 = \lambda_1$, λ_2 or λ_3 because $\lambda_4 > 1 - \lambda_0 - \lambda_1$. Assume that $\alpha_3 = \lambda_1$, then we get trigonometric equation

$$\frac{\sin\theta_n \sin 5\theta_n}{2\cos\theta_n \sin 2\theta_n \sin 3\theta_n} = \frac{\sin 2\theta_n}{2\sin\theta_n \sin 3\theta_n}.$$

Solving this equation, we obtain $n = 10$. Similarly $\alpha_3 = \lambda_2$ (resp. $\alpha_3 = \lambda_3$) implies $n = 16$ (resp. $n = 28$).

For any of the above data $(n ; \alpha_1, \cdots, \alpha_m)$, there exists a family of extended Jones' projections, or we have the following existence theorem.

Theorem 2. *Let M be a type II_1-factor. Then for everyone of data $(n ; \lambda_k, \lambda_{n-k-2})$ with $0 \leq k \leq \left[\frac{n-2}{2}\right]$, $(2k ; \lambda_0, \lambda_0, \lambda_{k-2})$ with $k \geq 2$, $(10 ; \lambda_0, \lambda_1, \lambda_1)$, $(16 ; \lambda_0, \lambda_1, \lambda_2)$ or $(28 ; \lambda_0, \lambda_1, \lambda_3)$ there exists a family of extended Jones' projections corresponding to it.*

Actually we construct a family of extended Jones' projections by use of string algebra, as explained below.

Let G be an unoriented pointed graph. Moreover we require that G be bipartite, locally finite and accessible. Denote a distinguished point by $*$.

Definition 2 (cf. [3]). For $x, y \in G^{(0)}$, $n \in \boldsymbol{N}$, we put

$Path_x^{(n)}$ = *the set of paths of length n with source x,*

$Path_{x,y}^{(n)} = \{\xi \in Path_x^{(n)} ;\ r(\xi) = y\}$,

$String_x^{(n)}$ = *the set of strings of length n with source x,*

H_n = *Hilbert space with orthonormal basis $Path_*^{(n)}$.*

For a string $\rho = (\rho_+, \rho_-) \in String_*^{(n)}$, we represent ρ on H_n by $\rho\xi = \delta(\rho_-, \xi)\rho_+$ for $\xi \in H_n$, and denote by A_n a finite dimensional C^*-algebra generated by $String_*^{(n)}$.

Let μ be a weight which is a map $G^{(0)} \to \boldsymbol{R}^+ = \{\lambda ; \lambda > 0\}$ with $\mu(*) = 1$, and Λ be Laplacian of G. Assume that μ is harmonic i.e. $\Lambda\mu = \beta\mu$ with $\beta \in \boldsymbol{R}^+$ and define a trace tr on A_n by $tr(\rho) = \beta^{-n}\mu(r(\rho))\delta(\rho_+, \rho_-)$ for $\rho = (\rho_+, \rho_-) \in String_*^{(n)}$. For $n \in \boldsymbol{N}$ a projection $e_n \in A_{n+1}$ is defined by

$$e_n=\beta^{-1}\sum_{\alpha\in Path_*^{(n-1)}}\sum_{\xi,\eta\in Path^{(1)}_{r(\alpha)}}\frac{\sqrt{\mu(r(\xi))\mu(r(\eta))}}{\mu(r(\alpha))}(\alpha\circ\xi\circ\xi^{\sim},\alpha\circ\eta\circ\eta^{\sim})\in A_{n+1}.$$

Then it can be proved by calculations that the sequence $\{e_n\,;n=1,2,\cdots\}$ satisfies the following relations (cf. [3]):

(a) $e_ne_{n\pm1}e_n=\beta^{-2}e_n$ for $n\in\boldsymbol{N}$; (b) $e_ne_m=e_me_n$ for $|m-n|\geq 2$;

(c) $tr(\omega e_{m+1})=\beta^{-2}tr(\omega)$ for any word ω in $e_1,\ldots,e_m$.

Moreover for an $x\in G^{(0)}$ such that $Path^{(1)}_{*,x}\neq\emptyset$, we define a projection $f_x\in A_1$ by $f_x=\sum_{\xi\in Path^{(1)}_{*,x}}(\xi,\xi)$. Then the next proposition gives the relations between f_x and e_n.

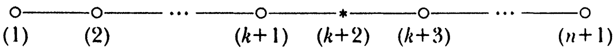
Proposition 1. (1) $e_1f_xe_1=\#(Path^{(1)}_{*,x})\mu(x)\beta^{-1}e_1$,

(2) $f_xe_n=e_nf_x$ *for* $n\geq 2$.

Let us now construct a family of extended Jones' projections.

1) CASE OF $(n\,;\lambda_k,\lambda_{n-k-2})$: Let G be a Dynkin diagram of type A_{n+1} and the distinguished point $*$ be a vertex with distance $k+1$ from the end vertex.

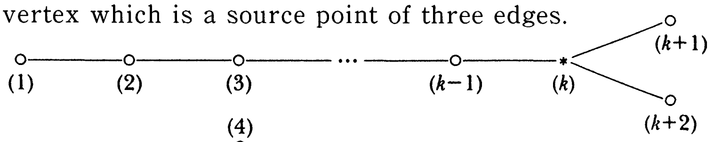
A_{n+1}



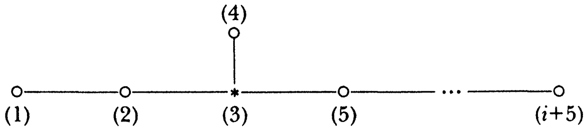
Then $\beta=2\cos(\pi/(n+2))$, $\mu((i))=\sin i\theta_n/\sin(k+2)\theta_n$. Take e_n, f_x with $x=(k+1)$, $(k+3)$, and denote $f_{(k+1)}$, $f_{(k+3)}$ by f_1, f_2 respectively. From [3] and Proposition 1, we see that $\{e_n,f_1,f_2\,;n\geq 1\}$ is a family of extended Jones' projections corresponding to $(n\,;\lambda_k,\lambda_{n-k-2})$.

2) CASE OF $(2k\,;\lambda_0,\lambda_0,\lambda_{k-2})$ or $(n\,;\lambda_0,\lambda_1,\lambda_i)(1\leq i\leq 3)$: Let G be a Dynkin diagram of type D_{k+2} or E_{i+5} respectively and the distinguished point $*$ be a vertex which is a source point of three edges.

D_{k+2}



E_{i+5}



Similarly we can construct a family of extended Jones' projections.

§2. The indicies of the pairs of II_1-factors. Here for a pair of type II_1-factors $A\supset B$ generated by a family of extended Jones' projections, we give index $[A:B]$ by using Wenzl's index formula.

Theorem 3. *Let M be a type II_1-factor, $\{e_i,f_j\,;i\geq 1,1\leq j\leq m\}$ be a family of extended Jones' projections in M corresponding to $(n\,;\alpha_1,\cdots,\alpha_m)$ and $A=\{e_i,f_j\,;i\geq1,1\leq j\leq m\}''$, $B=\{e_i\,;i\geq 1\}''$. Then A and B are hyperfinite type II_1-factors and the index $[A:B]$ is given as follows:*

1) *Case of* $(n\,;\alpha_1,\alpha_2)=(n\,;\lambda_k,\lambda_{n-k-2})\left(0\leq k\leq\left[\frac{n-2}{2}\right]\right)$:

$$[A:B]=\frac{\sin^2(k+2)\theta_n}{\sin^2\theta_n},\ \textit{with}\ \theta_n=\frac{\pi}{n+2}.$$

2) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(2k\,;\lambda_0,\lambda_0,\lambda_{k-2})(k\geq 2)$: $[A:B]=2\cot^2\theta_n$.

3) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(10\,;\lambda_0,\lambda_1,\lambda_1)$: $[A:B]=18+10\sqrt{3}$.

4) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(16\,;\lambda_0,\lambda_1,\lambda_2)$:

$$[A:B]=9\left\{2\sin^2\theta_n\left(\frac{\sin^2 2\theta_n}{\sin^2 4\theta_n}+\frac{\sin^2\theta_n}{\sin^2 3\theta_n}+1\right)\right\}^{-1}.$$

5) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(28\,;\lambda_0,\lambda_1,\lambda_3)$;

$$[A:B]=15\left\{2\sin^2\theta_n\left(\frac{\sin^2\theta_n}{\sin^2 5\theta_n}+\frac{\sin^2 3\theta_n}{\sin^2 5\theta_n}+\frac{\sin^2\theta_n}{\sin^2 3\theta_n}+1\right)\right\}^{-1}.$$

§3. Relative commutant $B'\cap A$.

Theorem 4. *Let M be a type II_1-factor, $\{e_i, f_j\,; i\geq 1, 1\leq j\leq m\}$ be a family of extended Jones' projections in M corresponding to $(n\,;\alpha_1,\cdots,\alpha_m)$ and $A=\{e_i\,;f_j\,;i\geq 1, 1\leq j\leq m\}''$, $B=\{e_i\,;i\geq 1\}''$. Then relative commutant $B'\cap A$ is trivial.*

Proof. Here we give the proof in case of $(n\,;\alpha_1,\alpha_2,)=(n\,;\lambda_k,\lambda_{n-k-2})\left(0\leq k\leq\left[\frac{n-2}{2}\right]\right)$. Other cases can be treated similarly.

Let G be a Dynkin diagram of type A_{n+1}, the distinguished point $*$ be a vertex with distance $k+1$ from the end vertex and $A(G)$ be a hyperfinite II_1-factor generated by string algebras of G. Then we can construct a family of extended Jones' projections $\{e_i, f_j\,;i\geq 1, 1\leq j\leq 2\}$ corresponding to$(n,\alpha_1,\alpha_2)=(n\,;\lambda_k,\lambda_{n-k-2})$ and put $A=\{e_i, f_j\,;i=1,2,\cdots,1\leq j\leq m\}''$ and $B=\{e_i\,;i=1,2,\cdots\}''$. From Theorem 3, we have $[A:B]=\sin^2(k+2)\theta_n/(\sin^2\theta_n)$. On the other hand, $[A(G):B]=\sin^2(k+2)\theta_n/(\sin^2\theta_n)$ by Prop. 4. 5. 2 of [1]. Since $A(G)\supset A\supset B$, we obtain $A(G)=A$. So by $A(G)\cap B'=\boldsymbol{C}$ it follows that $A\cap B'=\boldsymbol{C}$.

§4. Fixed point subalgebras for permutations of f_j's. For a family of extended Jones' projections $\{e_i, f_j\,;i\geq 1, 1\leq j\leq 3\}$, we define von Neumann subalgebras $A(j)$ of A $(j=1,2,3)$ by $A(j)=\{e_i, f_j\,;i\geq 1\}''$. Since $\{e_i, f_j\,;1-f_j\,;i\geq 1\}$ is a family of extended Jones' projections corresponding to $(n\,;\alpha_j, 1-\alpha_j)$, we have, by Theorem 3, that $A(j)$ is a hyperfinite II_1-factor and $[A(j):B]=\sin^2(k_j+2)\theta_n/(\sin^2\theta_n)$, where k_j is an integer such that $\lambda_{k_j}=\alpha_j$.

Since $[A:B]=[A:A(j)][A(j):B]$, the next theorem follows by Theorem 3 and a simple calculation.

Theorem 5. *Let A and $A(j)$ be as above. Then index for a pair $A\supset A(j)$ is given as follows.*

1) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(2k\,;\lambda_0,\lambda_0,\lambda_{k-2})$ $(k\geq 2)$:
$[A:A(1)]=[A:A(2)]=(2\sin^2\theta_n)^{-1}$, $[A:A(3)]=2$.

2) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(10\,;\lambda_0,\lambda_1,\lambda_1)$:
$[A:A(1)]=6+2\sqrt{3}$, $[A:A(2)]=[A:A(3)]=3+\sqrt{3}$.

3) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(16\,;\lambda_0,\lambda_1,\lambda_2)$:
$[A:A(j)]=9\beta\{2\sin^2(j+1)\theta_n\}^{-1}$ $(j=1,2,3)$,

where $\beta^{-1}=\dfrac{\sin^2 2\theta_n}{\sin^2 4\theta_n}+\dfrac{\sin^2\theta_n}{\sin^2 3\theta_n}+1.$

4) *Case of* $(n\,;\alpha_1,\alpha_2,\alpha_3)=(28\,;\lambda_0,\lambda_1,\lambda_3)$:
$[A:A(j)]=15\gamma\{2\sin^2(k_j+2)\theta_n\}^{-1}$ $(j=1,2,3)$,

where $\gamma^{-1} = \dfrac{\sin^2\theta_n}{\sin^2 5\theta_n} + \dfrac{\sin^2 3\theta_n}{\sin^2 5\theta_n} + \dfrac{\sin^2\theta_n}{\sin^2 3\theta_n} + 1$ *and* $(k_1, k_2, k_3) = (0,1,3)$.

Now let us consider automorphisms of A by permutations of $\{f_j ; 1 \le j \le m\}$. If $\sigma \in \mathrm{Aut}(A)$ and $\sigma(f_i) = f_j$, then $tr(f_j) = tr(\sigma(f_i)) = tr(f_i)$ i.e. $\alpha_i = \alpha_j$. So there exists such an automorphism, if and only if

$(n ; \alpha_1, \alpha_2) = (2k ; \lambda_{k-1}, \lambda_{k-1})$ with $k \ge 1$, or

$(n ; \alpha_1, \alpha_2, \alpha_3) = (2k ; \lambda_0, \lambda_0, \lambda_{k-2})$ with $k \ge 2$, or $(10 ; \lambda_0, \lambda_1, \lambda_1)$.

Here we consider fixed point algebras in case of $(n ; \alpha_1, \alpha_2) = (2k ; \lambda_{k-1}, \lambda_{k-1})$ with $k \ge 2$ and $(n ; \alpha_1, \alpha_2, \alpha_3) = (10 ; \lambda_0, \lambda_1, \lambda_1)$.

1) CASE OF $(n ; \alpha_1, \alpha_2) = (2k ; \lambda_{k-1}, \lambda_{k-1})$ for $k \ge 2$: Take $\sigma \in \mathrm{Aut}(A)$ such that $\sigma(f_1) = f_2$, $\sigma(f_2) = f_1$ and $\sigma(e_i) = e_i$ for $i \ge 1$. Since $A \supset A^\sigma \supset B$ and $B' \cap A = \boldsymbol{C}$, σ is an outer automorphism of A. Hence $[A : A^\sigma] = |\langle\sigma\rangle| = 2$. On the other hand, $[A : B] = (\sin^2\theta_n)^{-1}$ from Theorem 3. Since $[A : B] = (\sin^2\theta_n)^{-1} \ne 2 = [A : A^\sigma]$, we have $A^\sigma \supsetneqq B$ and $[A^\sigma : B] = (2\sin^2\theta_n)^{-1}$. It follows that $(A^\sigma)' \cap A = \boldsymbol{C}$ from $B' \cap A = \boldsymbol{C}$.

2) CASE OF $(n ; \alpha_1, \alpha_2, \alpha_3) = (10 ; \lambda_0, \lambda_1, \lambda_1)$: Define $\sigma \in \mathrm{Aut}(A)$ by $\sigma(f_1) = f_1$, $\sigma(f_2) = f_3$, $\sigma(f_3) = f_2$ and $\sigma(e_i) = e_i$ for $i \ge 1$. Comparing indicies, we have $A^\sigma \supsetneqq A(1)$ and $[A^\sigma : A(1)] = 3 + \sqrt{3}$.

From the above arguments, we obtain the next theorem.

Theorem 6. *Notations are as above.*

1) *Case of* $(n ; \alpha_1, \alpha_2) = (2k ; \lambda_{k-1}, \lambda_{k-1})$ *with* $k \ge 2$:

$$A^{S_2} \supsetneqq B,\ [A^{S_2} : B] = (2\sin^2\theta_n)^{-1},\ B' \cap A^{S_2} = \boldsymbol{C}.$$

2) *Case of* $(n ; \alpha_1, \alpha_2) = (10 ; \lambda_0, \lambda_1, \lambda_1)$:

$$A^{S_2} \supsetneqq A(1),\ [A^{S_2} : A(1)] = 3 + \sqrt{3},\ A(1)' \cap A^{S_2} = \boldsymbol{C}.$$

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