46. *Singular Variation of Domain and Eigenvalues of the Laplacian with the Third Boundary Condition*

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1. Introduction. This paper is a continuation of previous paper [6].

Let Ω be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary $\partial\Omega$. Let $\tilde{w}$ be a fixed point in Ω. Let $B(\varepsilon, \tilde{w})$ be the disk of radius ε with the center $\tilde{w}$. we put $\Omega_\varepsilon = \Omega\backslash\overline{B(\varepsilon, \tilde{w})}$. Consider the following eigenvalue problem

(1.1)
$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \qquad x \in \Omega_\varepsilon \\ u(x) &= 0 \qquad x \in \partial\Omega \\ u(x) + k\varepsilon^\sigma \frac{\partial u}{\partial \nu_x}(x) &= 0 \qquad x \in \partial B(\varepsilon, \tilde{w}). \end{aligned}$$

Here k denotes a positive constant. And σ is a real number. Here $\frac{\partial}{\partial \nu_x}$ denotes the derivative along the exterior normal direction with respect to Ω_ε.

Let $\mu_j(\varepsilon) > 0$ be the j-th eigenvalue of (1.1). Let μ_j be the j-th eigenvalue of the problem

(1.2)
$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \qquad x \in \Omega \\ u(x) &= 0 \qquad x \in \partial\Omega. \end{aligned}$$

Main aim of this paper is to give the following theorems. The details of our proof of theorems will be published elsewhere.

Let $\varphi_j(x)$ be the L^2-normalized eigenfunction associated with μ_j. We have the following.

Theorem 1. *Assume that μ_j is a simple eigenvalue. Then,*
$$\mu_j(\varepsilon) = \mu_j - 2\pi\, \varphi_j(\tilde{w})^2/(\log \varepsilon) + \mathbf{O}(|\log \varepsilon|^{-2}),$$
for $\sigma \geqq 1$.

Theorem 2. *Assume that μ_j is a simple eigenvalue. Then,*
$$\begin{aligned} \mu_j(\varepsilon) &= \mu_j + Q_j\varepsilon^{1-\sigma} + R_j\varepsilon^2 + \mathbf{O}(\varepsilon^{2-\sigma}) && (-1 < \sigma < 0) \\ \mu_j(\varepsilon) &= \mu_j + R_j\varepsilon^2 + Q_j\varepsilon^{1-\sigma} + \mathbf{O}(\varepsilon^3 |\log \varepsilon|) && (-2 < \sigma \leqq -1) \\ \mu_j(\varepsilon) &= \mu_j + R_j\varepsilon^2 + \mathbf{O}(\varepsilon^3 |\log \varepsilon|) && (\sigma \leqq -2), \end{aligned}$$
where
$$\begin{aligned} Q_j &= (2\pi/k)\, \varphi_j(\tilde{w})^2 \\ R_j &= -\pi(2\,|\operatorname{grad} \varphi_j(\tilde{w})|^2 - \mu_j\varphi_j(\tilde{w})^2). \end{aligned}$$

Remark. The case $\sigma \in [0, 1)$ is treated in [6]. It is curious to the authors that the asymptotic behaviour of $\mu_j(\varepsilon) - \mu_j$ is the same when $\sigma \leqq -2$. For the related papers we have Ozawa [7]-[9], Rauch-Taylor [10], Besson [3], Chavel [4] and the references in the above papers.

For other related problems on singular variation of domains the readers may be referred to Anné [1], Arrieta, Hale, and Han [2], Jimbo [5].

2. Outline of the proof of Theorems 1 and 2. Let $G(x, y)$ (resp. $G_\varepsilon(x, y)$) be the Green function of the Laplacian in Ω (resp. Ω_ε) associated with boundary condition (1.2) (resp. (1.1)).

We introduce the following kernel $p_\varepsilon(x, y)$.

$$(2.1)\qquad p_\varepsilon(x, y) = G(x, y) + g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) + h(\varepsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y)\rangle + i(\varepsilon)\langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y)\rangle,$$

where

$$\langle \nabla_w u(\tilde{w}), \nabla_w v(\tilde{w})\rangle = \sum_{n=1}^{2} \frac{\partial u}{\partial w_n}\frac{\partial v}{\partial w_n}\Big|_{w=\tilde{w}}$$

$$\langle H_w u(\tilde{w}), H_w v(\tilde{w})\rangle = \sum_{m,n=1}^{2} \frac{\partial^2 u}{\partial w_m \partial w_n}\frac{\partial^2 v}{\partial w_m \partial w_n}\Big|_{w=\tilde{w}}$$

when $w = (w_1, w_2)$ is an orthonormal frame of $\boldsymbol{R}^2$. Here $g(\varepsilon)$, $h(\varepsilon)$, $i(\varepsilon)$ are determined so that

$$(2.2)\qquad p_\varepsilon(x, y) + k\varepsilon^\sigma \frac{\partial}{\partial \nu_x} p_\varepsilon(x, y) \qquad x \in \partial B(\varepsilon, \tilde{w})$$

is small in some sense.

If we put

$$(2.3)\qquad g(\varepsilon) = -(\gamma - (2\pi)^{-1}\log\varepsilon + k(2\pi)^{-1}\varepsilon^{\sigma-1})^{-1}$$

$$(2.4)\qquad h(\varepsilon) = (k\varepsilon^\sigma - \varepsilon)/((2\pi\varepsilon)^{-1} + k(2\pi)^{-1}\varepsilon^{\sigma-2}) \quad (\sigma < 0)$$
$$= 0 \quad (\sigma \geqq 1)$$

and

$$(2.5)\qquad i(\varepsilon) = k\varepsilon^{\sigma+1}/(\pi^{-1}\varepsilon^{-2} + 2k\pi^{-1}\varepsilon^{\sigma-3}) \quad (\sigma < 0)$$
$$= 0 \quad (\sigma \geqq 1),$$

the above aim for (2.2) to be small is attained. Here

$$\gamma = \lim_{x \to \tilde{w}} (G(x, \tilde{w}) + (2\pi)^{-1}\log|x - \tilde{w}|).$$

We put

$$(\boldsymbol{G}f)(x) = \int_\Omega G(x, y)f(y)dy$$

$$(\boldsymbol{G}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x, y)f(y)dy$$

and

$$(\boldsymbol{P}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} p_\varepsilon(x, y)f(y)dy \quad (\sigma < 0)$$
$$= \int_\Omega p_\varepsilon(x, y)f(y)dy \quad (\sigma \geqq 0).$$

In case of $\sigma < 0$, $\boldsymbol{P}_\varepsilon$ cannot operate on $L^p(\Omega)$ because of the existence of $h(\varepsilon)$-term and $i(\varepsilon)$-term in (2.1).

Let T and T_ε be operators on Ω and Ω_ε, respectively. Then, $\| T \|_p$, $\| T_\varepsilon \|_{p,\varepsilon}$ denotes the operator norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively. Let f and f_ε be functions on Ω and Ω_ε, respectively. Then, $\| f \|_p$, $\| f_\varepsilon \|_{p,\varepsilon}$ denotes the norm on $L^p(\Omega)$, $L^p(\Omega_\varepsilon)$, respectively.

At first we outline the proof of Theorem 1. A crucial part of our proof of Theorem 1 is the following.

Theorem 3. *Fix $\sigma \geqq 1$. Then, there exists a constant C such that*

(2.6) $$\| \chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon - \boldsymbol{G}_\varepsilon \|_{2,\varepsilon} \leqq C\varepsilon |\log \varepsilon|^{-1}$$

holds.

Here χ_ε is the characteristic function of $\bar{\Omega}_\varepsilon$.

Since $\boldsymbol{G}_\varepsilon$ is approximated by $\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon$ and the difference between $\boldsymbol{P}_\varepsilon$ and $\chi_\varepsilon \boldsymbol{P}_\varepsilon \chi_\varepsilon$ is small in some sense, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G} \to \boldsymbol{P}_\varepsilon$. This is the outline of our proof of Theorem 1.

Next we outline the proof of Theorem 2. One important part of our proof of Theorem 2 is the following.

Theorem 4. *Fix $\sigma < 0$. Then, there exists a constant C such that*

(2.7) $$\| (\boldsymbol{P}_\varepsilon - \boldsymbol{G}_\varepsilon)(\chi_\varepsilon \varphi_j) \|_{2,\varepsilon} \leqq C\varepsilon^{2-\sigma} \quad (-1 < \sigma < 0)$$
$$\leqq C\varepsilon^3 |\log \varepsilon| \quad (\sigma \leqq -1)$$

hold.

We fix j and put

(2.8) $$\begin{aligned} p_\varepsilon(x, y) = G(x, y) &- \pi \mu_j \varepsilon^2 \cdot G(x, \tilde{w}) G(\tilde{w}, y) \\ &+ g(\varepsilon) G(x, \tilde{w}) G(\tilde{w}, y) \\ &+ h(\varepsilon) \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \xi_\varepsilon(x) \xi_\varepsilon(y) \\ &+ i(\varepsilon) \langle H_w G(x, \tilde{w}), H_w G(\tilde{w}, y) \rangle \xi_\varepsilon(x) \xi_\varepsilon(y), \end{aligned}$$

where $\xi_\varepsilon(x) \in C^\infty(\boldsymbol{R}^2)$ satisfies $|\xi_\varepsilon(x)| \leqq 1$, $\xi_\varepsilon(x) = 1$ for $x \in \boldsymbol{R}^2 \setminus \overline{B(\varepsilon, \tilde{w})}$, $\xi_\varepsilon(x) = 0$ for $x \in B(\varepsilon/2, \tilde{w})$ and $\xi_\varepsilon(x - \tilde{w})$ is rotationary invariant. Furthermore we put

$$(\bar{\boldsymbol{P}}_\varepsilon f)(x) = \int_\Omega p_\varepsilon(x, \mathrm{y}) f(\mathrm{y}) d\mathrm{y}.$$

The other important part of our proof of Theorem 2 is the follwing.

Theorem 5. *Fix $\sigma < 0$. Then, there exists a constant C such that*

(2.9) $$\| (\chi_\varepsilon \boldsymbol{P}_\varepsilon - \boldsymbol{P}_\varepsilon \chi_\varepsilon) \varphi_j \|_{2,\varepsilon} \leqq C\varepsilon^{2-\sigma} \quad (-1 < \sigma < 0)$$
$$\leqq C\varepsilon^3 |\log \varepsilon| \quad (\sigma \leqq -1)$$

hold.

Since (2.7) and (2.9) are both $o(\varepsilon^2)$, we know that everything reduces to our investigation of the perturbative analysis of $\boldsymbol{G} \to \bar{\boldsymbol{P}}_\varepsilon$. This is the outline of our proof of Theorem 2.

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