45. *Majorizations and Quasi-Subordinations for Certain Analytic Functions*

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Two subclasses $A(\alpha, \beta)$ and $R(p)$ of certain analytic functions in the open unit disk U are introduced. For these classes a majorization problem and a quasi-subordination problem of analytic functions in U are discussed.

1. Introduction. Let $A(\alpha, \beta)$ be the class of functions of the form

$$h(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \qquad (c_n \geqq 0) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and satisfy

$$\mathrm{Re}\{h(z) + \alpha z h'(z)\} > \beta \qquad (z \in U), \tag{1.2}$$

where $\mathrm{Re}(\alpha) \geqq 0$ and $0 \leqq \beta < 1$. The class $A(\alpha, \beta)$ for real $\alpha \geqq 0$ was studied by Altintas [1].

Also, let $R(p)$ denote the class of functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.3}$$

which are analytic in U and satisfy

$$\mathrm{Re}\left\{\sqrt[p]{\frac{g(z)}{s(z)}}\right\} > \frac{1}{2} \qquad (z \in U) \tag{1.4}$$

for some function $s(z)$ is analytic and univalent in U with $s(0) = 0$ and $s'(0) = 1$, where $p \geqq 1$.

Let $f(z)$ and $g(z)$ be analytic in U. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in U satisfying $w(0) = 0$, $|w(z)| \leqq |z|$ $(z \in U)$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad (z \in U) \quad \text{(cf. [4, p. 226])}. \tag{1.5}$$

Further, $f(z)$ is said to be quasi-subordinate to $g(z)$ if there exists an analytic function $w(z)$ such that $f(z)/w(z)$ is analytic in U,

$$\frac{f(z)}{w(z)} \prec g(z) \quad (z \in U), \tag{1.6}$$

and $|w(z)| \leqq 1$ $(z \in U)$. We also denote this quasi-subordination by

$$f(z) \underset{q}{\prec} g(z) \qquad (z \in U). \tag{1.7}$$

Note that the quasi-subordination (1.7) is equivalent to

$$f(z) = w(z) g(\phi(z)), \tag{1.8}$$

where $|w(z)| \leqq 1$ $(z \in U)$ and $|\phi(z)| \leqq |z|$ $(z \in U)$ (cf. [5]).

In the quasi-subordination (1.7), if $w(z) \equiv 1$, then (1.7) becomes the subordination (1.5).

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For analytic functions $f(z)$ and $g(z)$ in U, we say that $f(z)$ is majorized by $g(z)$ if there exists an analytic function $w(z)$ in U satisfying $|w(z)| \leqq 1$ and $f(z) = w(z)g(z)$ $(z \in U)$. We denote this majorization by

$$f(z) \ll g(z) \qquad (z \in U) \quad \text{(cf. [3])}. \tag{1.9}$$

If we take $\phi(z) = z$ in (1.8), then the quasi-subordination (1.7) becomes the majorization (1.9).

2. A majorization problem. To complete the proof of our result for majorization, we need the following lemmas.

Lemma 1. *If* $h(z)$ *defined by* (1.1) *is in the class* $A(\alpha, \beta)$, *then*

$$\sum_{n=1}^{\infty} c_n \leqq \frac{1+\beta}{1+\mathrm{Re}(\alpha)}. \tag{2.1}$$

Proof. We note that $h(z) \in A(\alpha, \beta)$ gives

$$\mathrm{Re}(1 - \sum_{n=1}^{\infty} (1+\alpha n)\, c_n z^n\} > \beta \qquad (z \in U). \tag{2.2}$$

Letting $z \rightarrow 1^-$ along the real axis, we find that

$$\sum_{n=1}^{\infty} (1 + n\mathrm{Re}(\alpha))\, c_n \leqq 1 - \beta, \tag{2.3}$$

and, since $c_n \geqq 0$ and $\mathrm{Re}(\alpha) \geqq 0$,

$$(1+\mathrm{Re}(\alpha)) \sum_{n=1}^{\infty} c_n \leqq \sum_{n=1}^{\infty} (1 + n\mathrm{Re}(\alpha))\, c_n \leqq 1 - \beta. \tag{2.4}$$

This gives the coefficient inequality (2.1).

Lemma 2. *If* $h(z)$ *defined by* (1.1) *is in the class* $A(\alpha, \beta)$, *then*

$$1 - \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z| \leqq \mathrm{Re}(h(z)) \leqq |h(z)| \leqq 1 + \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z| \tag{2.5}$$

for $z \in U$.

Proof. Since

$$|h(z)| \leqq 1 + |z| \sum_{n=1}^{\infty} c_n, \tag{2.6}$$

Lemma 1 leads to

$$|h(z)| \leqq 1 + \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z|. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} \mathrm{Re}(h(z)) &= 1 - \mathrm{Re}\left\{\sum_{n=1}^{\infty} c_n z^n\right\} \geqq 1 - \left|\sum_{n=1}^{\infty} c_n z^n\right| \\ &\geqq 1 - |z| \sum_{n=1}^{\infty} c_n \geqq 1 - \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z|. \end{aligned} \tag{2.8}$$

Now we prove

Theorem 1. *Let* $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_1 \neq 0,\ a_n \geqq 0)$ *be analytic in* U. *If* $f(z) \ll g(z)$ *and* $zg'(z)/g(z) \in A(\alpha, \beta)$, *then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r(\alpha, \beta)), \tag{2.9}$$

where $r(\alpha, \beta)$ *is the root of the cubic equation*

$$\begin{aligned} &(1-\beta)r^3 - (1+\mathrm{Re}(\alpha))r^2 \\ &+ (\beta - 2\mathrm{Re}(\alpha) - 3)r + 1 + \mathrm{Re}(\alpha) = 0 \end{aligned} \tag{2.10}$$

contained in the interval (0, 1).

Proof. For $g(z)$ such that $zg'(z)/g(z) \in A(\alpha, \beta)$, we have from Lemma 2 that

(2.11) $$\left|\frac{zg'(z)}{g(z)}\right| \geqq 1 - \frac{1-\beta}{1+\mathrm{Re}(\alpha)}r \qquad (|z| = r),$$

or

(2.12) $$|g(z)| \leqq \frac{(1+Re(\alpha))r}{1+\mathrm{Re}(\alpha)-(1-\beta)r}|g'(z)| \qquad (|z| = r).$$

Since $f(z) \ll g(z)$, there exists an analytic function $w(z)$ such that $f(z) = w(z)g(z)$ and $|w(z)| \leqq 1$ $(z \in U)$. Thus we have

(2.13) $$f'(z) = w(z)g'(z) + w'(z)g(z).$$

Noting that $w(z)$ satisfies

(2.14) $$|w'(z)| \leqq \frac{1-|w(z)|^2}{1-|z|^2} \qquad (z \in U) \qquad \text{(cf. [4, p. 168])},$$

we see that

(2.15) $$\begin{aligned} &|f'(z)| \\ &\leqq \left\{|w(z)| + \frac{1-|w(z)|^2}{1-r^2}\frac{(1+\mathrm{Re}(\alpha))r}{1+\mathrm{Re}(\alpha)-(1-\beta)r}\right\}|g'(z)| \\ &= \frac{-(1+\mathrm{Re}(\alpha))rX^2 + (1-r^2)(1+\mathrm{Re}(\alpha)-(1-\beta)r)X + (1+\mathrm{Re}(\alpha))r}{(1-r^2)(1+\mathrm{Re}(\alpha)-(1-\beta)r)} \\ &\qquad \times |g'(z)|, \end{aligned}$$

where $X = |w(z)|$. Note that the function $H(X)$ defined by

$$\begin{aligned} H(X) = &-(1+\mathrm{Re}(\alpha))rX^2 + (1-r^2)(1+\mathrm{Re}(\alpha)-(1-\beta)r)X \\ &+ (1+\mathrm{Re}(\alpha))r \quad (0 \leqq X \leqq 1) \end{aligned}$$

takes its maximum value at $X = 1$ with the condition (2.10). Let $r(\alpha, \beta)$ $(0 < r(\alpha, \beta) < 1)$ be the root of the equation (2.10). If $0 \leqq a \leqq r(\alpha, \beta)$, then the function

(2.16) $$\begin{aligned} \phi(X) = &-(1+\mathrm{Re}(\alpha))aX^2 \\ &+ (1-a^2)(1+\mathrm{Re}(\alpha)-(1-\beta)a)X + (1+\mathrm{Re}(\alpha))a \end{aligned}$$

increases in the interval $0 \leqq X \leqq 1$ so that $\phi(X)$ does not exceed $\phi(1) = (1-a^2)(1+\mathrm{Re}(\alpha)-(1-\beta)a)$. Therefore, from this fact, (2.15) gives the inequality (2.9).

3. A quasi-subordination problem. Our result of quasi-subordinations for the class $R(p)$ contained in

Theorem 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* U *and* $g(z) \in R(p)$. *If* $f(z) \underset{q}{\prec} g(z)$, *then*

(3.1) $$|a_n| \leqq \frac{(p+n)!}{(p+1)!(n-1)!} \qquad (n \geqq 2).$$

Equality in (3.1) *is attained for the function* $f(z)$ *given by*

(3.2) $$f(z) = \frac{z}{(1-z)^{p+2}}.$$

Proof. It follows from $f(z) \underset{q}{\prec} g(z)$ that

(3.3) $$f(z) \stackrel{q}{=} w(z)g(\phi(z)),$$

where $w(z)$ is analytic in U with $|w(z)| \leqq 1$ $(z \in U)$ and $\phi(z)$ is analytic in U with $|\phi(z)| \leqq |z|$ $(z \in U)$. Define the function $h(z)$ by

(3.4) $$\sqrt[p]{\frac{g(z)}{s(z)}} = h(z)$$

with a function $s(z)$ analytic and univalent in U. Then $g(z) \in R(p)$ gives

$\mathrm{Re}(h(z)) > 1/2$ $(z \in U)$, that is,

$$h(z) \prec \frac{1}{1-z} \qquad (z \in U). \tag{3.5}$$

Let

$$H = \{h(z) : h(z) \prec 1/(1-z)\}$$

and

$$H^p = \{h(z)^p : h(z) \in H\}.$$

Then from [6, p. 16], we have

$$\mathrm{exc\bar{o}}\, H^p = \{u(z) : u(z) = 1/(1-\eta z)^p, \ |\eta| = 1\}, \tag{3.6}$$

where $\mathrm{exc\bar{o}}\, H^p$ means the set of extreme points of the closed convex hull of H^p. If we take $g(z)/s(z) = k(z)$, then we have

$$f(z) = w(z)s(\phi(z))k(\phi(z)). \tag{3.7}$$

Letting $Q(z) = w(z)s(\phi(z))$ and $R(z) = k(\phi(z))$, we get

$$Q(z) \underset{q}{\prec} s(z) \quad (z \in U) \tag{3.8}$$

and

$$R(z) \prec k(z) \quad (z \in U). \tag{3.9}$$

Since $Q(z)$ is of the form $\sum_{n=1}^{\infty} q_n z^n$ and $s(z)$ is analytic and univalent in U, using [5] and [2], we have $|q_n| \leqq n (n \geqq 2)$.

Noting that $k(z) = h(z)^p \in H^p$, it follows from (3.9) that $R(z)$ is subordinate to a function belonging to $\mathrm{exc\bar{o}}\, H^p$. This gives

$$R(z) \prec \frac{1}{(1-\eta z)^p} \qquad (z \in U\,;\, |\eta| = 1) \tag{3.10}$$

or

$$R(z) \prec \frac{1}{(1-\eta\phi(z))^p} \qquad (|\eta| = 1). \tag{3.11}$$

Without a loss of generality, we can take $\eta = 1$, so $R(z) = 1/(1-\phi(z))^p$. Since

$$\frac{1}{1-\phi(z)} \prec \frac{1}{1-z} \qquad (z \in U),$$

the modulus of every coefficient of $1/(1-\phi(z))^p$ does not exceed the corresponding coefficient of $1/(1-z)^p$ (cf. [6, p. 17]). Therefore, we have

$$|r_n| \leqq \frac{(p+n-1)!}{n!(p-1)!}, \tag{3.12}$$

where $R(z) = 1 + r_1 z + r_2 z^2 \ldots.$ Since

$$\begin{aligned} f(z) &= Q(z)R(z) \\ &= (q_1 z + q_2 z^2 + \ldots)(1 + r_1 z + r_2 z^2 + \ldots), \end{aligned} \tag{3.13}$$

that is,

$$a_n = q_n + q_{n-1} r_1 + q_{n-2} r_2 + \ldots + q_1 r_{n-1}, \tag{3.14}$$

with the help of $|q_n| \leqq n$ $(n \leqq 2)$ and (3.12), we obtain

$$\begin{aligned} |a_n| &\leqq n + (n-1)p + (n-2)\frac{p(p+1)}{2!} \\ &\qquad + \ldots + \frac{p(p+1)\ldots(p+n-2)}{(n-1)!} \\ &= \frac{(p+n)!}{(p+1)!(n-1)!}. \end{aligned} \tag{3.15}$$

Finally, for the equality in (3.1), taking $w(z) = 1$, $\phi(z) = z$, $s(z) = z/(1 - z)^2$, and $k(z) = 1/(1 - z)^p$ in (3.7), we get $f(z) = z/(1 - z)^{p+2}$. This completes the assertion of Theorem 2.

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