5. *On Fundamental Units of Real Quadratic Fields with Norm +1*

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1. In our previous paper [2], we gave a new explicit form of the fundamental units of real quadratic fields with norm -1. In this note, we shall show that similar results also hold for the fundamental units of real quadratic fields with norm $+1$.

Let m be a positive integer which is not a perfect square and K be the real quadratic field $\boldsymbol{Q}(\sqrt{m})$. ε_0 denotes the fundamental unit of K. N denotes the norm map from K to $\boldsymbol{Q}$, and for any $x \in K$, $\bar{x}$ will denote the conjugate of x. We put

$R_+ = \{K$: *real quadratic fields with* $N\,\varepsilon_0 = +1\}$,

$E_+ = \{\varepsilon$: *units of real quadratic fields such that* $N\varepsilon = +1$ *and* $\varepsilon + \bar{\varepsilon} \geq 3\}$.

Then it is easy to see $R_+ \subset \{\boldsymbol{Q}(\sqrt{a^2+4a}) : a \in \boldsymbol{N}\}$, where $\boldsymbol{N}$ is the set of all the natural numbers.

Fix now a unit $\varepsilon = (t+2+u\sqrt{m})/2 = (t+2+\sqrt{t(t+4)})/2 \in E_+$ $(t>0)$ for a while, and denote $\varepsilon^n = (t_n+2+u_n\sqrt{m})/2$.

Since $t_n + 2 = \varepsilon^n + \bar{\varepsilon}^n$, we have

$$\begin{aligned} t_{n+1} &= \varepsilon^{n+1} + \bar{\varepsilon}^{n+1} - 2 = (\varepsilon+\bar{\varepsilon})(\varepsilon^n + \bar{\varepsilon}^n) - \varepsilon^{n-1} - \bar{\varepsilon}^{n-1} - 2 \\ &= (t+2)(t_n+2) - (t_{n-1}+2) - 2 = (t+2)t_n - t_{n-1} + 2t \quad (n \geq 2). \end{aligned}$$

Using the fact $t_1 = t$ and $t_2 = t^2 + 4t$ and this recurrence, we get inductively $t \mid t_n$ and $t_{n+1} - t_n = (t+1)t_n - t_{n-1} + 2t > (t+1)(t_n - t_{n-1})$ $(n \geq 2)$. Hence $t_{n+1} - t_n \geq t(t+3)(t+1)^{n-1}$ $(n \geq 1)$. Furthermore we have

$$\begin{aligned} (t_{n+1} - t_n)^2 &= \{(\varepsilon^{n+1} + \bar{\varepsilon}^{n+1}) - (\varepsilon^n + \bar{\varepsilon}^n)\}^2 = (\varepsilon^{2n+2} + \bar{\varepsilon}^{2n+2}) + (\varepsilon^{2n} + \bar{\varepsilon}^{2n}) - 2(\varepsilon^{2n+1} + \bar{\varepsilon}^{2n+1}) \\ &\quad - 2(\varepsilon + \bar{\varepsilon}) + 4 = t_{2n+2} + 2 + t_{2n} + 2 - 2(t_{2n+1} + 2) - 2(t+2) + 4 = t t_{2n+1}. \end{aligned}$$

Therefore we have obtained the following lemma.

Lemma 1. *With the above notation, we have*

(i) $t_1 = t$, $t_2 = t^2 + 4t$, *and* $t_{n+1} = (t+2)t_n - t_{n-1} + 2t$ $(n \geq 2)$,

(ii) $t \mid t_n$ *and* $t_{n+1} - t_n \geq t(t+3)(t+1)^{n-1}$ $(n \geq 1)$,

(iii) $t t_{2n+1} = (t_{n+1} - t_n)^2$ $(n \geq 1)$.

Until now ε has been fixed. Now let ε vary in E_+ and write $t_n(\varepsilon) = \varepsilon^n + \bar{\varepsilon}^n - 2$.

Lemma 2. *For any* $\varepsilon \in E_+$ *and* $n \geq 2$, $t_n(\varepsilon)$ *is not a prime except in the case* $n = 2$ *and* $\varepsilon = (3+\sqrt{5})/2$.

Proof. Suppose n decomposes into $n = ij$, where $i, j \geq 2$. Then, from (ii) of Lemma 1, $\varepsilon^n = (\varepsilon^i)^j$ implies $t_i(\varepsilon) \mid t_n(\varepsilon)$, and furthermore $t_i(\varepsilon) \geq t_2(\varepsilon) \geq 5t$, and $t_n(\varepsilon) \geq 5t_i(\varepsilon)$. Hence $t_n(\varepsilon)$ is not prime in this case.

Next, suppose $n \geq 2$ and $t(\varepsilon) = t \geq 2$. Then one gets $t(\varepsilon) \mid t_n(\varepsilon)$ and $t_n(\varepsilon) \geq$

$t_2(\varepsilon) \geq t(\varepsilon)(t(\varepsilon)+4)$. Hence $t_n(\varepsilon)$ is also not prime in this case.

Finally, Lemma 1 (iii) implies $t_{2n+1}(\varepsilon)$ is not prime for any ε and $n \geq 1$. Hence $t_n(\varepsilon)$ $(n \geq 2)$ is prime if and only if $t(\varepsilon)=1$ and $n=2$, that is, $t_2(\varepsilon)=5$, which completes the proof.

Proposition 1. *Let $s \in \boldsymbol{N}$ and put $\varepsilon=(s+2+\sqrt{s(s+4)})/2$. If $\boldsymbol{Q}(\sqrt{s(s+4)}) \in R_+$, then ε is the fundamental unit of $\boldsymbol{Q}(\sqrt{s(s+4)})$ if and only if there exist no units η in this field such that $t_n(\eta)=s$. If $\boldsymbol{Q}(\sqrt{s(s+4)}) \notin R_+$, then $t_n(\eta) \neq s$ for any $\eta \in E_+$, $n \geq 2$ implies that s is a perfect square.*

Proof. The first part of this proposition is easy to see.

Assume now $\boldsymbol{Q}(\sqrt{s(s+4)}) \notin R_+$, that is, $\boldsymbol{Q}(\sqrt{s(s+4)})$ contains the fundamental unit ε_0 with norm -1. Then ε_0 is expressed in the form $\varepsilon_0=(r+\sqrt{r^2+4})/2$ $(r \in \boldsymbol{N})$ and $\varepsilon_0^2=(r^2+2+r\sqrt{r^2+4})/2$. If there exist no units $\eta \in E_+$ such that $t_n(\eta)=s$ for some $n \geq 2$, then we have $\varepsilon=\varepsilon_0^2$. Therefore $s=r^2$.

Conversely if $s=r^2$ for some $r \in \boldsymbol{N}$, then $\varepsilon=\varepsilon_0^2$ holds for $\varepsilon_0=(r+\sqrt{r^2+4})/2$.

Combining Lemma 2 and Proposition 1, we have the following

Theorem 1. *For any prime $p \neq 5$, $\varepsilon=(p+2+\sqrt{p(p+4)})/2$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{p(p+4)})$.*

One can easily generalize this theorem in the follwing way.

Proposition 2. *Let k be a given positive integer and $\varepsilon=(t+2+\sqrt{t(t+4)})/2$ be a unit. Then there exist only finitely many t and $n \geq 2$ such that $t_n(\varepsilon)=kp$ (p: prime).*

Proof. Assume $t_n(\varepsilon)=kp$ $(n \geq 2)$. Then $p \mid t$ or $t \mid k$. First we consider the case $p \mid t$. Then t is expressed in the form $t=pt_1$, where t_1 is a natural number. From the assumption, we have $kp=t_n(\varepsilon) \geq t_2(\varepsilon)=t(t+4)=pt_1(pt_1+4)$. Hence $k \geq t_1(pt_1+4)$. Hence there exist only finitely many primes p and natural numbers t_1. Therefore there exist only finitely many t in this case.

For the case $t \mid k$, it is obvious that there exist only finitely many t. Therefore there exist only finitely many t such that $t_n(\varepsilon)=kp$ (p: prime).

Next we shall show that for any fixed k and t, there exist only finitely many n such that $t_n(\varepsilon)=kp$ for some prime p. We put $n=l(2j+1)$, where $l=2^r$ $(r \geq 0$ and $j \geq 0)$. If $j \neq 0$, then we put $\eta=\varepsilon^l=(t(\eta)+2+\sqrt{t(\eta)(t(\eta)+4)})/2$. Using Lemma 1 (iii), $kp=t_{2j+1}(\eta)=t(\eta)(t_{j+1}(\eta)-t_j(\eta))^2$ implies $(t_{j+1}(\eta)-t_j(\eta)) \mid k$. Since $\lim_{j \to \infty}(t_{j+1}(\eta)-t_j(\eta))=\infty$ for any $t(\eta)$, there exist only finitely many $j \neq 0$ such that $(t_{j+1}(\eta)-t_j(\eta)) \mid k$.

Finally we put $\rho=\varepsilon^{2j+1}=(t(\rho)+2+\sqrt{t(\rho)(t(\rho)+4)})/2$. $\Omega(x)$ denotes the number of primes which divide x. Since $kp=t_l(\rho)$, we have $\Omega(t_l(\rho))=\Omega(k)+1$. On the other hand, $t_2(\rho)=t(\rho)(t(\rho)+4)$ implies $\Omega(t_l(\rho)) \geq \Omega(t_{l/2}(\rho))+1 \geq r$. Therefore r is bounded. Hence we have shown there exist only finitely many n such that $t_n(\varepsilon)=kp$ for some prime p.

From Propositions 1 and 2, we have shown the following

Theorem 2. *Let k be a given positive integer. For almost all p, $\varepsilon=(kp+2+\sqrt{kp(kp+4)})/2$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{kp(kp+4)})$.*

For the case $k=2$, we have the following

Corollary. *For any prime* $p\neq 2$, $\varepsilon=p+1+\sqrt{p(p+2)}$ *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{p(p+2)})$.

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