42. *Product Formula for Twisted MacPherson Classes*

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Introduction. The topological Euler characteristic χ is multiplicative, i.e., $\chi(X \times Y) = \chi(X)\chi(Y)$. For a manifold X, a generalization of $\chi(X)$ to higher dimensional cohomology classes is the Chern cohomology class $c^*(X)$, which satisfies the cross-product formula $c^*(X \times Y) = c^*(X) \times c^*(Y)$. For a (possibly singular) compact complex algebraic variety X, a generalization of $\chi(X)$ to higher dimensional homology classes is the Schwartz-MacPherson homology class $c_*(X)$, which in the smooth case is just the Poincaré dual of the usual Chern cohomology class $c^*(X)$ and the 0-th component of which is equal to $\chi(X)$[1, 4, 5]. Very recently, in connection with lifting Schwartz-MacPherson classes to intersection homology [2], the first author [3] proved the product formula for Schwartz-MacPherson classes, i.e., $c_*(X \times Y) = c_*(X) \times c_*(Y)$. The second author [6, 7] defined the "twisted" MacPherson class $c_{t*}(X)$, which includes Schwartz-MacPherson class $c_*(X)$ as a special case, i.e., $c_{1*}(X) = c_*(X)$. The 0-th component of $c_{t*}(X)$ is the "stratified weighted" Euler characteristic $\chi^t(X)$, which is a degree-dim$X$ polynomial of t, involves Euler characteristic of sigularities also and equals to $\chi(X)$ when $t = 1$. In [7] the second author showed the multiplicativity of χ^t, i.e., $\chi^t(X \times Y) = \chi^t(X)\chi^t(Y)$. In this note, by strengthening and modifying the proof of [3] we show the product formula $c_{t*}(X \times Y) = c_{t*}(X) \times c_{t*}(Y)$, thus the product formulae $c_*(X \times Y) = c_*(X) \times c_*(Y)$ and $\chi^t(X \times Y) = \chi^t(X)\chi^t(Y)$ follow as special cases. More generally we show a product formula for the transformation c_{t*} acting on constructible functions with polynomial coefficients (Theorem 4).

1. Preliminaries. The varieties we consider are all (possibly singular) compact complex algebraic varieties. Let $\boldsymbol{F}$ be the constructible function covariant functor, where $\boldsymbol{F}(X)$ is the abelian group freely generated by characteristic functions $\mathbf{1}_W$ for subvarieties W of X. For a morphism $f : X \to Y$ the pushforward $f_* : \boldsymbol{F}(X) \to \boldsymbol{F}(Y)$ is defined by $(f_*\mathbf{1}_W)(y) := \chi(f^{-1}(y) \cap W)$. Let $H_*(\ ;\boldsymbol{Z})$ be the usual $\boldsymbol{Z}$-homology covariant functor. Deligne and Grothendieck conjectured and MacPherson [3] proved that *there exists a unique natural transformation* $c_* : \boldsymbol{F} \to H_*(\ ;\boldsymbol{Z})$ *satisfying the extra condition that* $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ *for any smooth* X. To construct the transformation c_*, MacPherson first observes that $\boldsymbol{F}(X)$ is also freely

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generated by local Euler obstruction functions $\boldsymbol{Eu_W}$ for all subvarieties W of X, and then for a variety X he defines the homomorphism $c_*: \boldsymbol{F}(X) \to H_*(X\,;\boldsymbol{Z})$ by $c_*(\sum_W n_W Eu_W) := \sum_W n_W \hat{c}(W)$, where $\hat{c}(W)$ is the Chern-Mather homology class of W. For a variety X $c_*(X) := c_*(\mathbf{1}_X)$ is called the *Schwartz-MacPherson class* (or *Chern-Schwartz-MacPherson class*) of X (cf. [1, 5]). By considering the map from X to a point, we can see that the 0-th component of $c_*(X)$ is nothing but the Euler characteristic $\chi(X)$. In [6, 7] the second author extended the above transformation c_* as follows. Let $\boldsymbol{F}^t$ be the "twisted" constructible function functor, where $\boldsymbol{F}^t(X) = \boldsymbol{F}(X) \otimes \boldsymbol{Z}[t]$ and the pushforward $f_*^t: \boldsymbol{F}^t(X) \to \boldsymbol{F}^t(Y)$ is defined by $f_*^t(Eu_W) := \sum_S n_S t^{\dim W - \dim S} Eu_S$, provided that $f_*(Eu_W) = \sum_S n_S Eu_S$. Then *there exists a unique natural transformation* $c_{t*}: \boldsymbol{F}^t \to H_*(\,;\boldsymbol{Z}[t])$ *satisfying the extra condition that* $c_{t*}(\mathbf{1}_X) = c_t^*(X) \cap [X]$ *for any smooth* X*, where* $c_t^*(X) := \sum_{i \geq 0} t^i c^i(X)$ *is the Chern polynomial of* X. Here $c^i(X)$ denotes the i-th component of the Chern cohomology class $c^*(X)$. For a variety X $c_{t*}(X) := c_{t*}(\mathbf{1}_X)$ is called the *twisted MacPherson class* of X. By considering the map from X to a point, we can see that the 0-th component of $c_{t*}(X)$ is just the stratified weighted Euler characterisitic $\chi^t(X)$ [7].

Notation. i) For any pair of constructible functions $\alpha \in \boldsymbol{F}(X)$ and $\beta \in \boldsymbol{F}(Y)$, $\alpha \times \beta \in \boldsymbol{F}(X \times Y)$ is defined by

$$(\alpha \times \beta)(x, y) := \alpha(x)\beta(y).$$

ii) For morphisms $f: X \to X'$ and $g: Y \to Y'$, $f \times g: X \times Y \to X' \times Y'$ is classically defined by

$$(f \times g)(x, y) = (f(x), g(y)).$$

2. Proof of product formula $\boldsymbol{c_{t*}(X \times Y) = c_{t*}(X) \times c_{t*}(Y)}$. First we observe the following key lemma

Lemma 1. *Let* $f: X \to X'$ *and* $g: Y \to Y'$ *be morphisms. Then for* $\boldsymbol{\alpha} \in \boldsymbol{F}(X)$ *and* $\boldsymbol{\beta} \in \boldsymbol{F}(Y)$

$$(f \times g)_*(\boldsymbol{\alpha} \times \boldsymbol{\beta}) = (f_*\boldsymbol{\alpha}) \times (g_*\boldsymbol{\beta}).$$

I.e., the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{F}(\mathrm{x}) \otimes \boldsymbol{F}(Y) & \xrightarrow{\ \times\ } & \boldsymbol{F}(X \times Y) \\ f_* \otimes g_* \downarrow & & \downarrow (f \times g)_* \\ \boldsymbol{F}(X') \otimes \boldsymbol{F}(Y') & \xrightarrow{\ \times\ } & \boldsymbol{F}(X' \times Y'). \end{array}$$

Proof. Let S, T be subvarieties of X and Y, respectively. Then by the multiplicativity of the topological Euler-Poincaré characteristic $\chi(A \times B) = \chi(A)\chi(B)$, it follows that

$$(f \times g)_*(\mathbf{1}_S \times \mathbf{1}_T) = (f \times g)_*(\mathbf{1}_{S \times T}) = f_*(\mathbf{1}_S) \times g_*(\mathbf{1}_T).$$

Indeed
$$\begin{aligned} ((f \times g)_*(\mathbf{1}_{S \times T}))(x', y') &= \chi((f \times g)^{-1}(x', y') \cap (S \times T)) \\ &= \chi((f^{-1}(x') \times g^{-1}(y')) \cap (S \times T)) \\ &= \chi((f^{-1}(x') \cap S) \times (g^{-1}(y') \cap T)) \\ &= \chi((f^{-1}(x') \cap S)\chi(g^{-1}(y') \cap T)) \\ &= (f_*(\mathbf{1}_S) \times g_*(\mathbf{1}_T))(x', y'). \end{aligned}$$

Let $\boldsymbol{\alpha} = \sum_S n_S \mathbf{1}_S$ and $\boldsymbol{\beta} = \sum_T n_T \mathbf{1}_T$. Then

$$(f \times g)_*(\boldsymbol{\alpha} \times \boldsymbol{\beta}) = (f \times g)_*((\textstyle\sum_S n_S \mathbf{1}_S) \times (\sum_T n_T \mathbf{1}_T))$$

$$\begin{aligned} &= (f\times g)_*(\textstyle\sum_{S,T} n_S n_T \mathbf{1}_{S\times T}) \\ &= \textstyle\sum_{S,T} n_S n_T (f\times g)_*(\mathbf{1}_{S\times T}) \\ &= \textstyle\sum_{S,T} n_S n_T (f_*(\mathbf{1}_S)\times g_*(\mathbf{1}_T)) \\ &= \textstyle\sum_S n_S f_*(\mathbf{1}_S)\times \sum_T n_T g_*(\mathbf{1}_T) \\ &= (f_*\alpha)\times(g_*\beta). \end{aligned}$$

Q.E.D.

Since local Euler obstruction functions are constructible, we have the following

Corollary 2. *Let $f: X\to X'$ and $g: Y\to Y'$ be morphisms, and let V and W be subvarieties of X and Y, respectively. Then*

$$(f\times g)_*(Eu_V\times Eu_W) = (f_*Eu_V)\times(g_*Eu_W).$$

Then, using the multiplicativity of local Euler obstruction[4], which is essential for the product formula, we can show the "twisted" version of the above corollary:

Corollary 3. *Let $f: X\to X'$ and $g: Y\to Y'$ be morphisms, and let V and W be subvarieties of X and Y, respectively. Then*

$$(f\times g)^t_*(Eu_V\times Eu_W) = (f^t_*Eu_V)\times(g^t_*Eu_W).$$

Proof. Let $f_*Eu_V = \sum_S n_S Eu_S$ and $g_*Eu_W = \sum_T n_T Eu_T$. Then by Corollary 2 and by the multiplicativity of local Euler obstruction $Eu_{A\times B} = Eu_A \times Eu_B$, we get

$$\begin{aligned} (f\times g)_*(Eu_{V\times W}) &= (f\times g)_*(Eu_V\times Eu_W) \\ &= (\textstyle\sum_S n_S Eu_S)\times(\sum_T n_T Eu_T) \\ &= \textstyle\sum_{S,T} n_S n_T (Eu_S\times Eu_T) \\ &= \textstyle\sum_{S,T} n_S n_T Eu_{S\times T}. \end{aligned}$$

Thus by the definition of the twisted pushforward of a constructible function, we have

$$\begin{aligned} (f\times g)^t_*(Eu_V\times Eu_W) &= (f\times g)^t_*(Eu_{V\times W}) \\ &= \textstyle\sum_{S,T} n_S n_T\, t^{\dim(V\times W)-\dim(S\times T)} Eu_{S\times T} \\ &= (\textstyle\sum_S n_S\, t^{\dim V-\dim S} Eu_S)\times(\sum_T n_T\, t^{\dim W-\dim T} Eu_T) \\ &= (f^t_*Eu_V)\times(g^t_*Eu_W). \end{aligned}$$

Q.E.D.

Theorem 4. *Let X and Y be varieties. Then for any $\alpha\in \boldsymbol{F}^t(X)$ and $\beta\in\boldsymbol{F}^t(Y)$*

$$c_{t*}(\alpha\times\beta) = c_{t*}(\alpha)\times c_{t*}(\beta).$$

I.e., the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{F}^t(X)\otimes\boldsymbol{F}^t(Y) & \xrightarrow{\times} & \boldsymbol{F}^t(X\times Y) \\ c_{t*}\otimes c_{t*}\downarrow & & \downarrow c_{t*} \\ H_*(X;\boldsymbol{Z}[t])\otimes H_*(Y;\boldsymbol{Z}[t]) & \xrightarrow{\times} & H_*(X\times Y;\boldsymbol{Z}[t]). \end{array}$$

In particular, letting $\alpha = \mathbf{1}_X$ and $\beta=\mathbf{1}_Y$, we get the product formula for the twisted MacPherson classes:

$$c_{t*}(X\times Y) = c_{t*}(X)\times c_{t*}(Y).$$

Proof. By the linearity of c_{t*} and by the bilinearity of $\times$, it suffices to show that for any two subvarieties V and W of X and Y, respectively,

(4.1) $$c_{t*}(Eu_V\times Eu_W) = c_{t*}(Eu_V)\times c_{t*}(Eu_W).$$

Observation (4.2). (i) If V and W are non-singular, then by the definition of cross-product the equality (4.1) holds. (ii) If $\dim V = \dim W = 0$, then (4.1) holds.

We shall therefore proceed to prove (4.1) by induction on $k = \dim V + \dim W$. Suppose that (4.1) holds for $\dim V + \dim W < k$, and we show that (4.1) holds for $\dim V + \dim W = k$. Let $p : \widetilde{V} \to V$ and $q : \widetilde{W} \to W$ be resolutions of singularities of V and W. Then we get

$$p_* Eu_{\widetilde{V}} = Eu_V + \textstyle\sum_S n_S En_S \quad \text{and} \quad q_* Eu_{\widetilde{W}} = Eu_W + \sum_T n_T Eu_T,$$

where S's and T's are in the singular locus of V and W, respectively. Hence, by the definition of the twisted pushforward we have

(4.3) $$p^t_* Eu_{\widetilde{V}} = Eu_V + \textstyle\sum_S n_S \, t^{\dim V - \dim S} Eu_S$$

and $$q^t_* Eu_{\widetilde{W}} = Eu_W + \textstyle\sum_T n_T \, t^{\dim W - \dim T} Eu_T.$$

Thus we have

(4.4) $$p^t_* Eu_{\widetilde{V}} \times q^t_* Eu_{\widetilde{W}} = Eu_V \times Eu_W + \textstyle\sum_S n_S \, t^{\dim V - \dim S} Eu_S \times Eu_W + \sum_T n_T \, t^{\dim W - \dim T} Eu_V \times Eu_T + \sum_{S,T} n_S n_T \, t^{\dim V + \dim W - \dim S - \dim T} Eu_S \times Eu_T.$$

Observation (4.5).

$$\begin{aligned} c_{t*}(p^t_* Eu_{\widetilde{V}} \times q^t_* Eu_{\widetilde{W}}) &= c_{t*}(p \times q)^t_* (Eu_{\widetilde{V}} \times Eu_{\widetilde{W}}) && \text{(by Corollary 3)} \\ &= (p \times q)_* c_{t*}(Eu_{\widetilde{V}} \times Eu_{\widetilde{W}}) && \text{(by the naturality of } c_{t*}) \\ &= (p \times q)_* (c_{t*}(Eu_{\widetilde{V}}) \times c_{t*}(Eu_{\widetilde{W}})) && \text{(by Observation (4.2))} \\ &= p_* c_{t*}(Eu_{\widetilde{V}}) \times q_* c_{t*}(Eu_{\widetilde{W}}) \\ &= c_{t*}(p^t_* Eu_{\widetilde{V}}) \times c_{t*}(q^t_* Eu_{\widetilde{W}}). && \text{(by the naturality of } c_{t*}) \end{aligned}$$

By (4.4)

$$\begin{aligned} c_{t*}(Eu_V \times Eu_W) = {} & c_{t*}(p^t_* Eu_{\widetilde{V}} \times q^t_* Eu_{\widetilde{W}}) - \textstyle\sum_S n_S \, t^{\dim V - \dim S} c_{t*}(Eu_S \times Eu_W) \\ & - \textstyle\sum_T n_T \, t^{\dim W - \dim T} c_{t*}(Eu_V \times Eu_T) \\ & - \textstyle\sum_{S,T} n_S n_T \, t^{\dim V + \dim W - \dim S - \dim T} c_{t*}(Eu_S \times Eu_T). \end{aligned}$$

Therefore, by substituting the expression from Observation (4.5) for the first summand and applying the induction hypothesis to the other three, we get:

$$\begin{aligned} c_{t*}(Eu_V \times Eu_W) &= c_{t*}(p^t_* Eu_{\widetilde{V}} - \textstyle\sum_S n_S \, t^{\dim V - \dim S} Eu_S) \times c_{t*}(q^t_* Eu_{\widetilde{W}} - \sum_T n_T \, t^{\dim W - \dim T} Eu_T) \\ &= c_{t*}(Eu_V) \times c_{t*}(Eu_W) && \text{(by (4.3)). Q.E.D.} \end{aligned}$$

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