41. *Equilibrium Vector Potentials in* R^3

By Hiroshi YAMAGUCHI

Faculty of Education, Shiga University

(Communicated by Kunihiko KODAIRA, M. J. A., Sept. 14, 1992)

1. In potential theory we are familiar with the notion of equilibrium potential which came from the condenser in electricity. In this note we shall introduce the notion of equilibrium vector potential motivated by the solenoid.

In the textbooks of electromagnetism we see the description about a solenoid: Let Σ be a 2-dimensional torus in R^3 given by $(b-r)^2+z^2=a^2$ $(0<a<b)$ where (r, θ, z) is the cylindrical coordinates of R^3. Denote by D the solid torus bounded by Σ, and put $D'=R^3-D\cup\Sigma$. We positively and symmetrically wind a coil L with electric current I around Σ, n times. Then the solenoid (Σ equipped with I) induces the static magnetic field $B_o(x)$ which is given by

$$B_o(x)=\begin{cases}\dfrac{nI}{2\pi}\left(\dfrac{-\sin\theta}{r}, \dfrac{\cos\theta}{r}, 0\right) & \text{for } x\in D\\ (0,0,0) & \text{for } x\in D'.\end{cases}$$

It does not seem rigorous mathematically, because $B_o(x)$ must be singular only on L, not on Σ. We shall make it clear and introduce the notions of equilibrium current density and equilibrium vector potential for any domain D in R^3 bounded by real analytic smooth surfaces.

2. First let $J(x)=(f_1(x), f_2(x), f_3(x))$ satisfy

(a) $f_i\in C_0^\infty(R^3)$ $(i=1,2,3)$; (b) $\operatorname{div} J(x)=0$ in R^3.

Then Jdv_x, where dv_x denotes the euclidean volume element of R^3, is called a *volume current density in* R^3. Let γ be a 1-cycle in R^3 which bounds a 2-dimensional surface Q. We set

$$J[\gamma]=\int_Q J(x)\cdot n_x\, dS_x$$

where n_x denotes the unit outer normal vector of Q at x, and dS_x, the euclidean surface area element of Q at x, and where $\cdot$ means the inner product. We say that $J[\gamma]$ is the *total current density of* Jdv_x *through* $[\gamma]$. We consider the following vector-valued integrals:

$$A_J(x)=\frac{1}{4\pi}\int_{R^3}\frac{J(y)}{\|x-y\|}dv_y \quad \text{for } x\in R^3,$$

$$B_J(x)=\operatorname{rot} A_J(x)=\frac{1}{4\pi}\int_{R^3}J(x)\times\frac{x-y}{\|x-y\|^3}dv_y \quad \text{for } x\in R^3$$

where $\times$ means the vector product. After Biot-Savart, $A_J(x)$ is called the *vector potential for* Jdv_x, and $B_J(x)$, the *magnetic field induced by* Jdv_x.

Next let $D\Subset R^3$ be a domain bounded by real analytic smooth surfaces Σ, and put $D'=R^3-D\cup\Sigma$. Let $J(x)=(f_1(x), f_2(x), f_3(x))$ satisfy

(a') $f_i\in C^\infty(\Sigma)$ $(i=1,2,3)$;

(b') There exists a sequence $\{J_n dv_x\}_{n=1,2,\dots}$ of volume current densities in R^3 such that $J_n dv_x \to J dS_x$ in the sense of distributions, where dS_x denotes the surface area element of Σ at x.

Then JdS_x is called a *surface current density on* Σ. For any 1-cycle γ in $R^3-\Sigma$, we set $J[\gamma]=\lim_{n\to\infty} J_n[\gamma]$, which is called the *total current density of* JdS_x *through* $[\gamma]$. We put

$$A_J(x)=\frac{1}{4\pi}\int_\Sigma \frac{J(y)}{\|x-y\|}dS_y \quad \text{for } x\in R^3,$$

$$B_J(x)=\operatorname{rot} A_J(x)=\frac{1}{4\pi}\int_\Sigma J(y)\times\frac{x-y}{\|x-y\|^3}dS_y \quad \text{for } x\in R^3-\Sigma.$$

$A_J(x)$ is called the *vector potential for* JdS_x, and $B_J(x)$, the *magnetic field induced by* JdS_x. Note that $A_J(x)$ is continuous in R^3, while $B_J(x)$ has a gap along Σ such that, for any $y\in\Sigma$,

$$\lim_{\substack{x\to y\\ x\in D}} B_J(x)-\lim_{\substack{x\to y\\ x\in D'}} B_J(x)=n_y\times J(y). \tag{2.1}$$

A surface current density JdS_x on Σ is called an *equilibrium current density on* Σ, if the magnetic field $B_J(x)$ is identically zero on D'. Then $A_J(x)$ is called the *equilibrium vector potential for* JdS_x *in* R^3. Our main result is the following

Theorem. *Let* $\{\gamma_i\}_{i=1,\dots,q}$ *be a* 1*-dimensional homology base of* D. *Then we have*

(1) *For a fixed* $i(1\le i\le q)$, *there exists a unique equilibrium current density* $J_i dS_x$ *on* Σ *such that* $J_i[\gamma_j]=\delta_{ij}$ *(Kronecker's delta);*

(2) *Any equilibrium current density on* Σ *is written by a linear combination of* $\{J_i dS_x\}_{i=1,\dots,q}$.

3. We shall give a sketch of the proof of Theorem. Let γ be a 1-cycle in D. By H. Weyl [2], we find a unique square integrable harmonic 2-form Ω_γ in D such that

$$\int_\gamma \omega=(\omega,*\Omega_\gamma)_D \text{ for all } C^\infty \text{ closed 1-forms } \omega \text{ on } D\cup\Sigma. \tag{3.1}$$

We call $*\Omega_\gamma$ the *reproducing* 1*-form for* (D,γ). By use of De Rahm's theorem in the real analytic category (H. Cartan [1]), we have

Lemma 3.1. *There exists a unique real analytic* 1*-form* A_γ *in a neighborhood* V *of* Σ *in* R^3 *such that*

(i) $dA_\gamma=\Omega_\gamma$ *in* $D\cap V$; (ii) $d*A_\gamma=0$ *in* V; (iii) $A_\gamma=0$ *on* Σ.

We call A_γ the *vector potential of* Ω_γ *with boundary values* 0 *in* V. We write $\Omega_\gamma=ady\wedge dz+bdz\wedge dx+cdx\wedge dy$ in D, and $A_\gamma=Adx+Bdy+Cdz$ in V. We define the following vector fields B_γ in $R^3-\Sigma$ and J_γ on Σ:

$$B_\gamma=\begin{cases}(a,b,c) \text{ in } D\\ (0,0,0) \text{ in } D'\end{cases} \quad\text{and}\quad J_\gamma(x)=\left(\frac{\partial A}{\partial n_x},\frac{\partial B}{\partial n_x},\frac{\partial C}{\partial n_x}\right) \text{ for } x\in\Sigma.$$

Using these notations we have

Lemma 3.2. $J_\gamma dS_x$ *is an equilibrium current density on* Σ *which induces* B_γ *as a magnetic field in* $R^3-\Sigma$.

For each $\gamma_i(1\le i\le q)$ we denote by $*\Omega_i$ the reproducing 1-form for (D,γ_i). Then, by (3.1), $\{\Omega_i\}_{i=1,\dots,q}$ are linearly independent. From this fact with Lemma 3.2 we obtain theorem.

The vector potential A_γ in Lemma 3.1 plays a central role in [3]. We return to the vector field $B_o(x)$ stated in Introduction. Let $\Omega_o(x) = \frac{-\sin\theta}{r}dy \wedge dz + \frac{\cos\theta}{r}dz \wedge dx$ on $D \cup \Sigma$ and consider the circle γ given by $\{r = b\}$ in the solid torus D. Then, by simple calculation we see that $*\Omega_o(x)$ is the reproducing 1-form for (D, γ) up to a multiplicative constant. It follows from theorem and (2.1) that $B_o(x)$ is the magnetic field induced by the equilibrium current density

$$J_o dS_x = \frac{1}{(2\pi a)^2}\left(\frac{z\cos\theta}{r}, \frac{z\sin\theta}{r}, \frac{b-r}{r}\right) \text{ on } \Sigma.$$

Lemma 3.2 also implies the following stability of equilibrium current densities for any Σ:

Corollary 3.1. *Let JdS_x be an equilibrium current density on Σ. Then*

$$\int_{R^3} \| B_J \|^2(x)dv_x = \mathrm{Min}\,\{ \int_{R^3} \| B_K \|^2(x)dv_x \mid KdS_x \in \mathcal{S}\}$$

where $\mathcal{S}$ is the family of all surface current densities KdS_x on Σ such that $K[\gamma_j] = J[\gamma_j]$ $(1 \leq j \leq q)$.

The details of this note will be published elsewhere.

References

[1] H. Cartan: Variétés analytique réeles et variétés analytiques complexes. Bull. Soc. Math. de France, **85**, 77–99 (1957).

[2] H. Weyl: The method of orthogonal projection in potential theory. Duke Math. J., **7**, 411–444 (1940).

[3] H. Yamaguchi: Variations of harmonic modules of domains in R^3 (in preparation).