40. *Analytic Zariski Decomposition*

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1. Introduction. Let X be a projective variety and let D be a Cartier divisor on X. The following problem is fundamental in algebraic geometry.

Problem 1. *Study the linear system* $|\nu D|$ *for* $\nu \geq 1$.

To this problem, there is a rather well developed theory in the case of dim $X = 1$. In the case of dim $X = 2$, in early 60-th, O. Zariski reduced this problem to the case that D is nef($=$ numerically semipositive) by using his famous Zariski decomposition ([4]).

Recently Fujita, Kawamata etc. generalized the concept of Zariski decompositions to the case of dim $X \geq 3$ ([1, 2]). The definition is as follows.

Definition 1. *Let X be a projective variety and let D be an $\boldsymbol{R}$-Cartier divisor on X. The expression*

$$D = P + N \, (P, N \in Div(X) \otimes \boldsymbol{R})$$

is called a Zariski decomposition of D, if the following conditions are satisfied.

1. *P is nef,*
2. *N is effective,*
3. $H^0(X, \mathcal{O}_X([\nu P])) \simeq H^0(X, \mathcal{O}_X([\nu D]))$ *holds for all* $\nu \in \boldsymbol{Z}_{\geq 0}$ *where* [ ]*'s denote the integral parts of the divisors.*

Although many useful applications of this decomposition have been known ([1, 2, 3]), as for the existence, very little has been known. There is the following (rather optimistic) conjecture.

Conjecture 1. *Let X be a normal projective variety and let D be a pseudoeffective $\boldsymbol{R}$-Cartier divisor on X. Then there exists a modification $f : Y \rightarrow X$ such that $f^* D$ admits a Zariski decomposition.*

In this paper, I would like to announce a "weak solution" to this conjecture. Details will be published elsewhere. In this paper, all algebraic varieties are defined over $\boldsymbol{C}$.

2. Statement of the results. Definition 2. *Let X be a normal projective variety and let D be a $\boldsymbol{R}$-Cartier divisor on X. D is called big if*

$$\kappa(D) := \limsup_{\nu \rightarrow +\infty} \frac{\log \dim H^0(X, \mathcal{O}_X([\nu D]))}{\log \nu} = \dim X$$

holds. D is called pseudoeffective, if for any ample divisor H, $D + \varepsilon H$ is big for every $\varepsilon > 0$.

Definition 3. *Let M be a complex manifold of dimension n and let $A_c^{p,q}(M)$ denote the space of $C^\infty(p, q)$ forms of compact support on M with usual Fréchet space structure. The dual space $D^{p,q}(M) := A_c^{n-p,n-q}(M)^*$ is called the space of (p, q)-currents on M. The linear operators $\partial : D^{p,q}(M) \rightarrow D^{p+1,q}(M)$ and $\bar{\partial}: D^{p,q}(M) \rightarrow D^{p,q+1}(M)$ is defined by*

$$\partial T(\varphi) = (-1)^{p+q+1} T(\partial\varphi),\ T \in D^{p,q}(M),\ \varphi \in A_c^{n-p-1,n-q}(M)$$

and

$$\bar{\partial} T(\varphi) = (-1)^{p+q+1} T(\bar{\partial}\varphi),\ T \in D^{p,q}(M),\ \varphi \in A_c^{n-p,n-q-1}(M).$$

We set $d = \partial + \bar{\partial}$. $T \in D^{p,q}(M)$ *is called closed if* $dT = 0$. $T \in D^{p,p}(M)$ *is called real if* $T(\varphi) = T(\bar{\varphi})$ *holds for all* $\varphi \in A_c^{n-p,n-p}(M)$. *A real current* (p, p)*-current* T *is called positive if* $(\sqrt{-1})^{p(n-p)} T(\eta \wedge \bar{\eta}) \geq 0$ *holds for all* $\eta \in A_c^{p,0}(M)$.

Since codimension p subvarieties are considered to be closed positive (p, p)-currents, closed positive (p, p)-currents are considered as a completion of the space of codimension p subvarieties with respect to the topology of currents. For an $\boldsymbol{R}$ divisor D on a smooth projective variety X. We denote the class of D in $H^2(X, \boldsymbol{R})$ by $c_1(D)$.

Definition 4. *Let* T *be a closed positive* (p, p)*-current on the open unit ball* $B(1)$ *in* C^n *with centre* O. *The Lelong number* $\Theta(T, O)$ *of* T *at* O *is defined by*

$$\Theta(T, O) = \lim_{r \downarrow 0} \frac{1}{\pi^{n-p} r^{2(n-p)}} T(\chi(r)\omega^{n-p}),$$

where $\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ *and* $\chi(r)$ *be the characteristic function of the open ball of radius* r *with centre* O *in* C^n.

It is well known that the Lelong number is invariant under coordinate changes. Hence we can define the Lelong number for a closed positive (p, p)-current on a complex manifold. It is well known that if a closed positive current T is defined by a codimension p-subvariety the Lelong number $\Theta(T, x)$ coincides the multiplicities of the subvariety at x.

We note that thanks to Hironaka resolution of singularities, to solve the conjecture, we can restrict ourselves to the case that X is smooth. Our theorem is stated as follows.

Theorem 1. *Let* X *be a smooth projective variety and let* D *be a big* $\boldsymbol{R}$*-Cartier divisor on* X. *Then there exists a closed positive* (1,1)*-current* T *such that*

1. T *represents* $c_1(D)$ *in* $H^2(X, \boldsymbol{R})$,
2. *For every modification* $f : Y \to X$, $\nu \in \boldsymbol{Z}_{\geq 0}$ *and* $y \in Y$,

$$mult_y B_S \mid f^*([\nu D]) \mid \geq \nu\, \Theta(f^*T, y)$$

and

$$\liminf_{\nu \to +\infty} \frac{1}{\nu} mult_y B_S \mid f^*([\nu D]) \mid = \Theta(f^*T, y)$$

hold.

We call T an Analytic Zariski decomposition(AZD) of D. The relation between Zariski decomposition and AZD is described by the following cororally and proposition.

Cororally 1. *Let* X *be a smooth projective variety and let* D *be a nef and big* $\boldsymbol{R}$ *divisor on* X. *Then* $c_1(D)$ *can be represented by a closed positive* (1,1)*-current* T *with* $\Theta(T) \equiv 0$.

Proposition 1. *Let* X *be a smooth projective variety and let* D *be an* $\boldsymbol{R}$ *divisor on* X *such that* $c_1(D)$ *can be represented by a closed positive* (1,1) *current* T

with $\Theta(T) \equiv 0$. *Then* D *is nef.*

Let X, D be as in Theorem 1. Suppose that there exists a modification $f: Y \to X$ such that there exists a Zariski decomposition $f^*D = P + N$ of f^*D on Y. Then by Cororally 1 there exists a closed positive (1,1) current S such that $c_1(P) = [S]$ and $\Theta(S) \equiv 0$. Then the push-forward $T = f_*(S + N)$ is a AZD of D. The main advantage of AZD is that we can consider the existence without changing the space by modifications.

3. Sketch of the proof of Theorem 1. Now I would like to sketch the proof of Theorem 1. Let X, D be as in Theorem 1. Let ω_∞ be a C^∞ closed real (1,1) form representing the class of D. Let ω_0 be a C^∞ Kähler form on X. We set

$$\omega_t = (1 - e^{-t})\omega_\infty + e^{-t}\omega_0.$$

Let Ω be a C^∞ volume form on X. Now we consider the following initial value problem.

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u \text{ on } X \times [0, T)$$

$$u = 0 \quad \text{on } X \times \{0\},$$

where $n = \dim X$ and T is the maximal existence time for the C^∞ solution u. By the standard implicit function theorem T is positive. Actually T is the maximal t such that the de Rham cohomology class of ω_t belongs to the Kähler cone of X. Now we have the following proposition.

Proposition 2. *There exists a nonempty Zariski open subset* U *such that the solution* $u: X \to [-\infty, \infty)$ *exists in* $L^1(X)$ *and* C^∞ *on* U.

Now we set

$$T = \lim_{t\to\infty} (\omega_t + \sqrt{-1}\partial\bar{\partial}u),$$

where $\partial\bar{\partial}$ is taken in the sense of current. Then we can verify that T is analytic Zariski decomposition of D by using Hörmander's L^2 estimates for $\bar{\partial}$ operator and C^0-estimate of u.

References

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