25. Eisenstein Series on Quaternion Half-space of Degree 2

By Shoyu NAGAOKA
Department of Mathematics, Kinki University

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1992)

1. Eisenstein series. Let $\boldsymbol{H}$ denote the skew field of real Hamiltonian quaternions with the canonical basis $e_1=1, e_2, e_3, e_4$. Let $Her(n,\boldsymbol{H})$ denote the real Jordan algebra consisting of all quaternion Hermitian $n\times n$ matrices and $Pos(n,\boldsymbol{H}):=\{Y\in Her(n,\boldsymbol{H})\,|\,Y>0\}$ the open subset of all positive definite matrices. Then *the quaternion half-space of degree* n is given by

$$\mathcal{H}(n,\boldsymbol{H}):=\{Z=X+iY\,|\,X\in Her(n,\boldsymbol{H}),\ Y\in Pos(n,\boldsymbol{H})\}\subset Her(n,\boldsymbol{H})\otimes_{\boldsymbol{R}}\boldsymbol{C}.$$

Set $J_n=\begin{pmatrix}0_n & E_n\\ -E_n & 0_n\end{pmatrix}$. The group

$$G_n:=\{M\in M(2n,\boldsymbol{H})\,|\,{}^t\overline{M}J_nM=qJ_n \text{ for some } q\in\boldsymbol{R}_+\}$$

acts on $\mathcal{H}(n,\boldsymbol{H})$ in the usual way. Given $Z\in\mathcal{H}(n,\boldsymbol{H})$ and $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_n$ with $n\times n$ blocks A, B, C, D set

$$M\langle Z\rangle:=(AZ+B)(CZ+D)^{-1}.$$

The Hurwitz order is denoted

$$\mathcal{O}=\boldsymbol{Z}e_0+\boldsymbol{Z}e_1+\boldsymbol{Z}e_2+\boldsymbol{Z}e_3,\qquad e_0=\frac{1}{2}(e_1+e_2+e_3+e_4)$$

(cf. [1], [4]).

The group

$$\Gamma_n:=\{M\in M(2n,\mathcal{O})\,|\,{}^t\overline{M}J_nM=J_n\}$$

is called *the modular group of quaternions of degree* n. Let $\Gamma_{n,\infty}$ denote the subgroup of Γ_n defined by

$$\Gamma_{n,\infty}:=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_n\,|\,C=0_n\right\}.$$

Given $A\in M(n,\boldsymbol{H})$, $A^\vee$ denotes the element of $M(2n,\boldsymbol{C})$ obtained by the representation of quaternions as complex 2×2 matrices and we define $\delta(A)=\det^{1/2}(A^\vee)$ (we take as $\delta(A)>0$ for $A\in Pos(n,\boldsymbol{H})$).

We define a kind of Eisenstein series on $\mathcal{H}(n,\boldsymbol{H})$ by

$$E_k^{(n)}(Z,s)=\delta(Y)^{s/2}\sum_{\left(\begin{smallmatrix}* & *\\ C & D\end{smallmatrix}\right)\in\Gamma_{n,\infty}\backslash\Gamma_n}|\delta(CZ+D)|^{-s}\delta(CZ+D)^{-k},$$

where $k\in\boldsymbol{Z}$, $(Z,s)\in\mathcal{H}(n,\boldsymbol{H})\times\boldsymbol{C}$ and $Z=X+iY$. It is known that this series is absolutely convergent if $\mathrm{Re}(s)+k>2(2n-1)$. Put, for $Y\in Pos(n,\boldsymbol{H})$, $H\in Her(n,\boldsymbol{H})$, and $(\alpha,\beta)\in\boldsymbol{C}^2$,

$$\xi^{(n)}(Y,H;\alpha,\beta)=\int_{Her(n,\boldsymbol{H})}\boldsymbol{e}(-\tau(H,V))\delta(V+iY)^{-\alpha}\delta(V-iY)^{-\beta}dV,$$

where τ denotes the reduced trace form, $\boldsymbol{e}(s)=\exp(2\pi is)$ for $s\in\boldsymbol{C}$, and dV is the Euclidean measure on $Her(n,\boldsymbol{H})$ by viewing it as $\boldsymbol{R}^n\times\boldsymbol{H}^{(n(n-1))/2}$ (cf. [8], (1.25)). This integral is convergent if $\mathrm{Re}(\alpha+\beta)>2(2n-1)-1$ and is represented by the generalized hypergeometric function (see Shimura [8]).

Let Λ_n be the dual lattice of $Her(n,\mathcal{O})$ with respect to τ. We define a singular series by

$$\alpha^{(n)}(s,H)=\sum_R \nu(R)^{-s}\boldsymbol{e}(\tau(H,R)),\qquad (s,H)\in \boldsymbol{C}\times\Lambda_n,$$

where R runs over all representatives of $Her(n,\boldsymbol{H_Q})/Her(n,\mathcal{O})$ $(\boldsymbol{H_Q}=\mathcal{O}\otimes_{\boldsymbol{Z}}\boldsymbol{Q})$ and $\nu(R)=|\delta(C)|$ with $R=C^{-1}D$, $\begin{pmatrix} * & * \\ C & D\end{pmatrix}\in\Gamma_n$ (cf. [9]). It is known that this series is absolutely convergent if $\mathrm{Re}(s)>2(2n-1)$ and has an infinite product expansion of the form

$$\alpha^{(n)}(s,H)=\prod_{p:\text{prime}}\alpha_p^{(n)}(s,H),$$
$$\alpha_p^{(n)}(s,H)=\sum_{\substack{R_p\\ \nu(R_p):\text{power of } p}}\nu(R_p)^{-s}\boldsymbol{e}(\tau(H,R_p)).$$

According to [9], we call $\alpha_p^{(n)}$ *the Siegel series* for $H\in\Lambda_n$.

Proposition 1. *$E_k^{(n)}(Z,s)$ has a Fourier expansion of the form*

$$E_k^{(n)}(Z,s)=\delta(Y)^{s/2}\Big\{1+\sum_{j=1}^{n}\sum_{H\in\Lambda_j}\sum_{\{Q\}}2^{(j(j-1))/2}\xi^{(j)}\Big(Y[Q],H\,;k+\frac{s}{2},\frac{s}{2}\Big)$$
$$\times\alpha^{(j)}(k+s,H)\boldsymbol{e}(\tau(H,X[Q]))\Big\}$$

where Q is an $\mathcal{O}$-integral $n\times j$ matrix which can be completed with $n-j$ columms to a unimodular matrix $(Q*)$ and runs through a set of representatives of the classes $\{Q\}=\{QU\mid U\in GL(j,\mathcal{O})\}$ and $Y[Q]={}^t\bar{Q}YQ\in Her(j,\boldsymbol{H})$ (cf. Maass [6], §18).

2. Siegel series of degree 2. In order to give an explicit formula for $\alpha_p^{(2)}(s,H)$, we introduce some notation. For $H\in\Lambda_2$, and prime p, we define integers a, b $(a,b\geqq 0)$ by

$$p^b\|\varepsilon(H):=\max\{q\in\boldsymbol{N}\mid q^{-1}H\in\Lambda_2\},\qquad p^a\|2\delta(H)\in\boldsymbol{Z}.$$

Theorem 1. *$\alpha_p^{(2)}(s,H)$ has the following expression.*

(1) *If* $\mathrm{rank}\,H=2$, *then we have*

$$\alpha_p^{(2)}(s,H)=(1-p^{-s})(1-p^{2-s})F_p(s,H),$$

where
$$F_p(s,H)=\sum_{l=0}^{b}p^{l(5-s)}\Big(\sum_{m=0}^{a-2l}p^{m(3-s)}\Big)\qquad \textit{if } p\neq 2,$$
$$\alpha_2^{(2)}(s,H)=(1-2^{-s})F_2(s,H),$$

where
$$F_2(s,H)=\sum_{l=0}^{b}2^{l(5-s)}\Big(\sum_{m=0}^{a-2l}2^{m(3-s)}-2^{4-s}\sum_{m=0}^{a-2l\cdot 2}2^{m(3-s)}\Big).$$

(2) *If* $\mathrm{rank}\,H=1$, *then we have*

$$\alpha_p^{(2)}(s,H)=(1-p^{-s})(1-p^{2-s})(1-p^{3-s})^{-1}\sum_{l=0}^{b}p^{l(5-s)}\qquad \textit{if } p\neq 2,$$
$$\alpha_2^{(2)}(s,H)=(1-2^{-s})(1-2^{4-s})(1-2^{3-s})^{-1}\sum_{l=0}^{b}2^{l(5-s)}.$$

(3)
$$\alpha_p^{(2)}(s,0_2)=(1-p^{-s})(1-p^{2-s})(1-p^{3-s})^{-1}(1-p^{5-s})^{-1}\qquad \textit{if } p\neq 2,$$
$$\alpha_2^{(2)}(s,0_2)=(1-2^{-s})(1-2^{4-s})(1-2^{3-s})^{-1}(1-2^{5-s})^{-1}.$$

These formulae are obtained by a similar argument as in [2] (see, also [4], [7]).

Corollary. (1) *If* $\operatorname{rank} H=2$, *then*

$$\alpha^{(2)}(s,H)=\zeta(s)^{-1}\zeta(s-2)^{-1}(1-2^{2-s})^{-1}F(s,H),$$

where

$$F(s,H)=\prod_p F_p(s,H).$$

Moreover $F(s,H)$ *satisfies a functional equation of the form*

$$F(s,H)=|2\delta(H)|^{3-s}F(6-s,H).$$

(2) *If* $\operatorname{rank} H=1$, *then*

$$\alpha^{(2)}(s,H)=\zeta(s)^{-1}\zeta(s-2)^{-1}\zeta(s-3)(1-2^{4-s})(1-2^{2-s})^{-1}\sigma_{5-s}(\varepsilon(H)).$$

(3) $\alpha^{(2)}(s,0_2)=\zeta(s)^{-1}\zeta(s-2)^{-1}\zeta(s-3)\zeta(s-5)(1-2^{2-s})^{-1}(1-2^{4-s})$.

Remark. (1) In the case $n=1$, $\alpha^{(1)}(s,h)$ $(h\in \boldsymbol{Z})$ is given by

$$\alpha^{(1)}(s,h)=\begin{cases}\zeta(s)^{-1}\sigma_{1-s}(h) & \text{if } h\neq 0\\ \zeta(s)^{-1}\zeta(s-1) & \text{if } h=0.\end{cases}$$

(2) Proposition 2 shows that $\alpha^{(2)}$ depends only on s, $\delta(H)$ and $\varepsilon(H)$. Especially, we have

$$\alpha^{(2)}(s,{}^tH)=\alpha^{(2)}(s,H).$$

For the function $\xi^{(n)}(Y,H;\alpha,\beta)$, we have

$$\xi^{(n)}({}^tY,{}^tH;\alpha,\beta)=\xi^{(n)}(Y,H;\alpha,\beta).$$

3. Functional equation. By the corollary of Theorem 1, we get the following result.

Theorem 2. *Set*

$$\Phi(Z,s)=2^{s/2}\frac{1-2^{2-s}}{s-4}\xi(s)\xi(s-2)E_0^{(2)}(Z,s),$$

where $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. *Then* $\Phi(Z,s)$ *can be continued as a meromorphic function in s and satisfies the following functional equation*

$$\Phi({}^tZ,6-s)=\Phi(Z,s).$$

4. Explicit formula of Fourier coefficient. As an application of Theorem 1, we get an explicit formula of the Fourier coefficient of holomorphic Eisenstein series $E_k^{(2)}(Z,0)$. From the analytic properties of hypergeometric functions, Siegel series and Epstein zeta functions, we know that $\lim_{s\to 0}E_k^{(2)}(Z,s)$ is holomorphic in Z if $k\geqq 4$ and it is a modular form of weigh k for Γ_2. Let

$$\lim_{s\to 0}E_k^{(2)}(Z,s)=\sum_{0\leqq H\in\Lambda_2}a_k(H)\boldsymbol{e}(\tau(H,Z))$$

be the Fourier expansion.

Theorem 3. *We assume that k is an even integer such that* $k\geqq 4$. *Then* $a_k(H)$ *is given by the following formula:*

$$a_k(H)=\begin{cases}1 & \textit{if } H=0_2\\ -\dfrac{2k}{B_k}\sigma_{k-1}(\varepsilon(H)) & \textit{if } \operatorname{rank} H=1\\ -\dfrac{4k(k-2)}{(2^{k-2}-1)B_k\cdot B_{k-2}}\displaystyle\sum_{d\mid\varepsilon(H)}d^{k-1}\{\sigma_{k-3}(2\delta(H)/d^2)-2^{k-2}\sigma_{k-3}(\delta(H)/2d^2)\} & \textit{if } \operatorname{rank} H=2,\end{cases}$$

where B_k *is the k-th Bernoulli number and we understand that* $\sigma_k(m)=0$ *if* $m\notin \boldsymbol{N}$.

Remark. In [5], Krieg proved this formula by a different method ([5], Theorem 3).

We consider the following theta series

$$\Theta(Z, S_H)=\sum_{X\in\mathcal{L}} e\left(\frac{1}{2}\tau(S_H[X], Z)\right), \qquad Z\in\mathcal{H}(2, \boldsymbol{H}),$$

where $\mathcal{L}=M(2, \mathcal{O})$ and

$$S_H=\begin{pmatrix} 2 & e_1+e_2 \\ e_1-e_2 & 2 \end{pmatrix} \qquad \text{(cf. [3], p. 114).}$$

This series is a generator of the space of modular forms of weight 4 and has a Fourier expansion of the form

$$\Theta(Z, S_H)=\sum_{0\leqq H\in\Lambda_2} A(S_H, 2H)e(\tau(H, Z))$$

$$A(S_H, 2H)=\sharp\{X\in M(2, \mathcal{O}) \mid S_H[X]=2H\}.$$

This shows that $A(S_H, 2H)=a_4(H)$.

Acknowledgements. After this work was completed, Prof. A. Krieg and Prof. W. L. Baily, Jr. informed the author that Dr. Kim proved our Theorem 2 in his thesis independently. The author would like to express his gratitude to Prof. Krieg and Prof. Baily for their helpful comments.

References

[1] A. Hurwitz: Vorlesungen über die Zahlentheorie der Quaternionen. Springer-Verlag (1919).
[2] G. Kaufhold: Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktionen 2. Grades. Math. Ann., **137**, 454–476 (1959).
[3] A. Krieg: Modular forms on half-spaces of quaternions. Lect. Notes in Math., vol. 1143, Springer-Verlag (1980).
[4] ——: The elementary divisor theory over the Hurwitz order of integral quaternions. Linear and Multilinear Algebra, **21**, 325–344 (1987).
[5] ——: The Maaß space and Hecke operators. Math. Z., **204**, 527–550 (1990).
[6] H. Maass: Siegel's modular forms and Dirichlet series. Lect. Notes in Math., vol. 216, Springer-Verlag (1971).
[7] S. Nagaoka: An explicit formula for Siegel series. Abh. Math. Sem. Univ. Hamburg, **59**, 235–267 (1989).
[8] G. Shimura: Confluent hypergeometric functions on tube domains. Math. Ann., **260**, 269–302 (1982).
[9] ——: On Eisenstein series. Duke Math. J., **50**, 412–476 (1983).