80. On the Starlikeness of the Bernardi Integral Operator

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Abstract: Denote by A the class of functions f analytic in the unit disc D and normalised so that $f(0)=f'(0)-1=0$. For $f\in A$ and $-1<c\leq 0$, let F_c be defined by $F_c(z)=\frac{(1+c)}{z^c}\int_0^z t^{c-1}f(t)dt$ for $z\in D$. We find estimates on β so that $\operatorname{Re} f'(z)>\beta$ will ensure the starlikeness of F_c.

Introduction. Denote by A the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$ and normalised so that $f(0)=f'(0)-1=0$. Let R be the subclass of A satisfying $\operatorname{Re} f'(z)>0$ for $z\in D$ and S^* be the subset of starlike functions, i.e.

$$S^*=\left\{f\in A:\ \operatorname{Re}\frac{zf'(z)}{f(z)}>0 \text{ for } z\in D\right\}.$$

If S denotes the subset of A consisting of univalent functions, then it is well known that $R\subset S$ and $S^*\subset S$. Krzyz [3] gave an example to show that R is not a subset of S^*. On the other hand, Singh and Singh [6] showed that $f\in R$ would imply $F_0\in S^*$, where

$$F_0(z)=\int_0^z \frac{f(t)}{t}\,dt.$$

In a later paper, Singh and Singh [7] showed that $\operatorname{Re} f'(z)>-\frac{1}{4}$ is sufficient to ensure $F_0\in S^*$ and more recently [5] it was shown that $\operatorname{Re} f'(z)>-0.262$ implies the same.

Suppose that $f\in A$ and $c>-1$. For $z\in D$, the Bernardi operator [1] is defined by

$$F_c(z)=\frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{1}$$

It was shown in [5] that $\operatorname{Re} f'(z)>\beta$ implies $F_c\in S^*$ provided

$$(1+c)\beta>\frac{\log(4/e)}{6}\left(c^2\tan^2\frac{\alpha^*\pi}{2}-3\right) \tag{2}$$

where $1=\alpha^*+(2/\pi)\tan^{-1}\alpha^*$. We note that when $c=0$, $\beta=-0.193$ which is not as good an estimate as the constant -0.262. If $c=1$, then $\beta=-0.017$, which was also obtained in [4].

In this paper we shall improve the constant β in (2) for $-1<c\leq 0$.

Results. Theorem. Suppose $f\in A$ and F_c be given by (1) and that

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$$S(1+c)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1+c}.$$

For $-1<c\leq 0$ and $z\in D$, let $\operatorname{Re}f'(z)>\beta$. Then $F_c\in S^*$ provided

$$\beta=\frac{3+2(1+c)[kS(1+c)-1]}{2(1+c)[kS(1+c)-1]}, \tag{3}$$

where

$$k=1-2c-\log\frac{4}{e}$$

and

$$1-3(1+c)^{-1}\left[2+\left(1-\log\frac{4}{e}\right)^2\right]^{-1}\leq\beta\leq 0.$$

We shall need the following lemma.

Lemma [5]. Let p be analytic in D with $p(0)=1$. Suppose that $\alpha>0$, $\beta<1$ and that for $z\in D$, $\operatorname{Re}(p(z)+\alpha zp'(z))>\beta$.

Then for $z\in D$,

$$\operatorname{Re}p(z)>1+2(1-\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\alpha n}.$$

Proof of Theorem. From (1) we have

$$F_c'(z)+\frac{1}{1+c}zF_c''(z)=f'(z), \tag{4}$$

and so using the lemma with $\alpha=1/(1+c)>0$, we obtain

$$\operatorname{Re}F_c'(z)>1+2(1-\beta)(1+c)S(1+c)=\mu \text{ say}. \tag{5}$$

Applying the lemma again with $\alpha=1$ and $\beta=\mu$ gives

$$\operatorname{Re}\frac{F_c(z)}{z}>(1-\mu)\log\frac{4}{e}+\mu. \tag{6}$$

Note also that from (4),

$$F_c'(z)+zF_c''(z)=(1+c)f'(z)-cF_c'(z),$$

and so since $c\leq 0$, the hypotheses of the theorem and (5) give

$$\operatorname{Re}\{zF_c''(z)+F_c'(z)\}>(1+c)\beta-c\mu. \tag{7}$$

We now use the Clunie-Jack lemma [2]. Let

$$\frac{zF_c'(z)}{F_c(z)}=\frac{1+w(z)}{1-w(z)},$$

so that w is analytic in D, $w(0)=0$ and $w(z)\neq 1$. Then

$$zF_c''(z)+F_c'(z)=\frac{F_c(z)}{z}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2+\frac{2zw'(z)}{(1-w(z))^2}\right]. \tag{8}$$

Thus we need to show that $|w(z)|<1$ for $z\in D$. Suppose that there exists $z_0\in D$ such that for $|z|\leq|z_0|$, $\max|w(z)|=|w(z_0)|=1$. Then the Clunie-Jack lemma implies that $z_0w'(z_0)=kw(z_0)=ke^{i\theta}$ for $0<\theta<2\pi$ and where $k\geq 1$. With $z=z_0$, it follows from (8) that

$$\begin{aligned}\operatorname{Re}\{z_0F_c''(z_0)+F_c'(z_0)\}&=\operatorname{Re}\left\{\frac{F_c(z_0)}{z_0}\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right]\right\}\\&\leq-\frac{k}{2\sin^2(\theta/2)}\operatorname{Re}\frac{F_c(z_0)}{z_0}.\end{aligned}$$

Since $(1-\mu)\log(4/e)+\mu>0$ and $k\geq 1$,

$$\operatorname{Re}\{z_0F_c''(z_0)+F_c'(z_0)\}\leq -\frac{1}{2}\left[(1-\mu)\log\frac{4}{e}+\mu\right]=(1+c)\beta-c\mu,$$

where we have used (6). Thus at $z=z_0$ we have a contradiction to (7) and so the result is proved.

Remarks. 1. When $c=0$, (3) reduces to $3\beta+(1-\beta)(2-\log(4/e))\log(4/e)=0$, which is the result in [5].

2. For $-1<c\leq 0$, the value of β in (3) is smaller than the value β in (2).

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