79. A Class of Inclusion Theorems Associated with Some Fractional Integral Operators[†]

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In the present paper the authors prove several inclusion theorems for some interesting subclasses of analytic functions involving a certain family of fractional integral operators. The corresponding results for the Hardy space $\mathcal{H}^p$ $(0<p\leqq\infty)$ follow as corollaries of these theorems. Some applications to the generalized hypergeometric functions are also considered.

1. Introduction. Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U}=\{z:|z|<1\}.$$

Definition 1. A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{R}(\gamma)$ if it satisfies the inequality:

$$\mathrm{Re}\{f'(z)\}>\gamma \quad (z\in\mathcal{U};\ 0\leqq\gamma<1).$$

The class $\mathcal{R}(0)$ was studied systematically by MacGregor [6] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part. Various interesting subclasses of $\mathcal{A}$ associated with the class $\mathcal{R}(\gamma)$ were considered elsewhere by (among others) Sarangi and Uralegaddi [11], Owa and Uralegaddi [8], and Srivastava and Owa [12].

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n, \tag{1.2}$$

and denote by $\mathcal{R}^*(\gamma)$ the class obtained by taking the intersection of the classes $\mathcal{R}(\gamma)$ and $\mathcal{T}$; that is,

$$\mathcal{R}^*(\gamma)=\mathcal{R}(\gamma)\cap\mathcal{T} \quad (0\leqq\gamma<1). \tag{1.3}$$

Finally, let $\mathcal{H}^p$ $(0<p\leqq\infty)$ denote the Hardy space of analytic functions $f(z)$ in $\mathcal{U}$, and define the integral means $M_p(r,f)$ by

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$$M_p(r,f)=\begin{cases}\left[\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right]^{1/p} & (0<p<\infty)\\ \max\limits_{|z|\leqq r}|f(z)| & (p=\infty).\end{cases} \tag{1.4}$$

Then, by definition, an analytic function $f(z)$ in $\mathcal{U}$ belongs to the Hardy space $\mathcal{H}^p$ $(0<p\leqq\infty)$ if

$$\lim_{r\to1-}\{M_p(r,f)\}<\infty \quad (0<p\leqq\infty). \tag{1.5}$$

For $1\leqq p\leqq\infty$, $\mathcal{H}^p$ is a Banach space with the norm defined by (cf. Duren [2, p. 23])

$$\|f\|_p=\lim_{r\to1-}M_p(r,f) \quad (1\leqq p\leqq\infty). \tag{1.6}$$

Furthermore, $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathcal{U}$, while $\mathcal{H}^2$ is the class of power series $\sum a_n z^n$ with $\sum|a_n|^2<\infty$.

The main object of the present paper is to prove some inclusion theorems for the classes $\mathcal{R}(\gamma)$ and $\mathcal{R}^*(\gamma)$ involving a certain family of fractional integral operators. As corollaries of these theorems, we derive the corresponding results for the Hardy space $\mathcal{H}^p$ $(0<p\leqq\infty)$. We also consider some relevant applications to the generalized hypergeometric functions.

2. Definitions and elementary properties of the fractional integral operators. Let λ_j $(j=1,\cdots,l)$ and μ_j $(j=1,\cdots,m)$ be complex numbers such that

$$\mu_j\neq0,\ -1,\ -2,\ \cdots \quad (j=1,\cdots,m).$$

Then the generalized hypergeometric function ${}_lF_m(z)$ is defined by (cf., e.g., [13, p. 333])

$$\begin{aligned}{}_lF_m(z)&\equiv{}_lF_m(\lambda_1,\cdots,\lambda_l;\mu_1,\cdots,\mu_m;z)\\ &=\sum_{n=0}^{\infty}\frac{(\lambda_1)_n\cdots(\lambda_l)_n}{(\mu_1)_n\cdots(\mu_m)_n}\frac{z^n}{n!} \quad (l\leqq m+1),\end{aligned} \tag{2.1}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in N=\{1,2,3,\cdots\}).\end{cases} \tag{2.2}$$

We note that the ${}_lF_m(z)$ series in (2.1) converges absolutely for $|z|<\infty$ if $l<m+1$, and for $z\in\mathcal{U}$ if $l=m+1$.

Making use of the Gaussian hypergeometric function which corresponds to (2.1) when $l-1=m=1$, Srivastava *et al.* [15] introduced the fractional integral operators $I_{0,z}^{\alpha,\beta,\eta}$ and $J_{0,z}^{\alpha,\beta,\eta}$ defined below (see also Owa *et al.* [9]).

Definition 2. For real numbers $\alpha>0$, β, and η, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}{}_2F_1\left[\alpha+\beta,-\eta;\alpha;1-\frac{\zeta}{z}\right]f(\zeta)\,d\zeta, \tag{2.3}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z)=O(|z|^\varepsilon) \quad (z\to0),$$

where

$$\varepsilon > \max\{0, \beta-\eta\}-1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

The operator $I_{0,z}^{\alpha,\beta,\eta}$ is a generalization of the fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ introduced by Saigo [10] and studied subsequently by Srivastava and Saigo [14] in connection with certain bounary value problems involving the celebrated Euler-Darboux equation.

Definition 3. Under the hypotheses of Definition 1, let

$$\alpha>0, \quad \min\{\alpha+\eta, -\beta+\eta, -\beta\}>-2, \quad \text{and} \quad 3 \geqq \frac{\beta(\alpha+\eta)}{\alpha}. \tag{2.4}$$

Then the fractional integral operator $J_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z). \tag{2.5}$$

In order to derive our main inclusion theorems, we shall also need the following

Lemma (cf. Srivastava *et al.* [15, p. 415, Lemma 3]). *Let α, β, η, and κ be real numbers.*

Then

$$I_{0,z}^{\alpha,\beta,\eta} z^{\kappa} = \frac{\Gamma(\kappa+1)\Gamma(\kappa-\beta+\eta+1)}{\Gamma(\kappa-\beta+1)\Gamma(\kappa+\alpha+\eta+1)} z^{\kappa-\beta} \quad (\alpha>0\,;\ \kappa>\beta-\eta-1). \tag{2.6}$$

3. Inclusion theorems. We begin by proving

Theorem 1. *Let the parameters α, β, and η satisfy the inequalities:*

$$\alpha>0, \quad \beta<0, \quad \textit{and} \quad \eta>\max\{\beta, -\alpha\}. \tag{3.1}$$

Suppose also that the function $f(z)$ defined by (1.2) *is in the class $\mathcal{R}^*(\gamma)$.*

Then

$$J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{R}^*(\gamma).$$

Proof. The hypothesis (3.1) readily implies the inequalities [cf. Equation (2.4)]

$$\min\{\alpha+\eta, -\beta+\eta, -\beta\}>0 \quad \text{and} \quad \frac{\beta(\alpha+\eta)}{\alpha}<0,$$

which obviously render the operator $J_{0,z}^{\alpha,\beta,\eta}$ well-defined.

Applying (2.2), (2.6), and Definition 3, we obtain

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = z - \sum_{n=2}^{\infty} \Phi(n)|a_n|z^n, \tag{3.2}$$

where, for convenience,

$$\Phi(n) = \frac{(2-\beta+\eta)_{n-1}(1)_n}{(2-\beta)_{n-1}(2+\alpha+\eta)_{n-1}} \quad (n \in N\backslash\{1\}). \tag{3.3}$$

Noting that $\Phi(n)$ is a non-decreasing function of n, we have

$$0<\Phi(n)\leqq\Phi(2)<1 \quad (n \in N\backslash\{1\}). \tag{3.4}$$

It follows from (3.2) and (3.3) that

$$J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{T}.$$

For a function $f(z) \in \mathcal{R}^*(\gamma)$, it is known that (cf. [11]; see also [8, p. 196, Lemma 2])

$$(3.5)\qquad \sum_{n=2}^{\infty} n|a_n| \leqq 1-\gamma,$$

which, in conjunction with (3.2) and (3.4), yields

$$\begin{aligned}\operatorname{Re}\{[J_{0,z}^{\alpha,\beta,\eta} f(z)]'\} &= 1-\operatorname{Re}\left\{\sum_{n=2}^{\infty} n\Phi(n)|a_n|z^{n-1}\right\}\\ &\geqq 1-\sum_{n=2}^{\infty} n\Phi(n)|a_n||z|^{n-1} > 1-\sum_{n=2}^{\infty} n|a_n|\\ &\geqq 1-(1-\gamma)=\gamma,\end{aligned}$$

whence $J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{R}^*(\gamma)$, completing the proof of Theorem 1.

Corollary 1. *Under the hypotheses of Theorem* 1,

$$f(z) \in \mathcal{H}^p \quad (0<p<\infty).$$

Proof. Corollary 1 follows easily from Theorem 1 by virtue of Lemma 3 of Jung *et al.* [3].

The proof of our next inclusion theorem would make use of the generalized Libera integral operator $\mathcal{J}_c$ defined by (cf. Owa and Srivastava [7]; see also [13, p. 338])

$$(3.6)\qquad \begin{aligned}\mathcal{J}_c f \equiv \mathcal{J}_c f(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt\\ &= z+\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n \quad (f \in \mathcal{A};\ c>-1).\end{aligned}$$

The operator $\mathcal{J}_c$ ($c \in N$) was introduced by Bernardi [1]. In particular, the operator $\mathcal{J}_1$ was studied earlier by Libera [4] and Livingston [5].

Making use of (3.6), we now prove

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{R}(\gamma)$. *If* $\alpha \in N$ *and* η *is unrestricted, in general, then*

$$J_{0,z}^{\alpha,-\alpha,\eta} f(z) \in \mathcal{R}(\gamma).$$

Proof. In terms of the Hadamard product (or convolution), we find from (3.6) and Definition 3 that

$$(3.7)\qquad \begin{aligned}J_{0,z}^{\alpha,-\alpha,\eta} f(z) &= z+\sum_{n=2}^{\infty}\frac{\alpha+1}{\alpha+n}\cdots\frac{1+1}{1+n}a_n z^n\\ &= \mathcal{J}_\alpha * \mathcal{J}_{\alpha-1} * \cdots * \mathcal{J}_1 f(z) \quad (\alpha \in N;\ \eta \text{ arbitrary}).\end{aligned}$$

Since [cf. Equation (3.6)]

$$(3.8)\qquad \mathcal{J}_c f = (c+1)\int_0^1 t^{c-1} f(zt)dt \quad (f \in \mathcal{A};\ c>-1),$$

we have

$$(3.9)\qquad \operatorname{Re}\left\{\frac{d}{dz}\mathcal{J}_c f(z)\right\} = (c+1)\int_0^1 t^c \operatorname{Re}\{f'(zt)\}dt \quad (f \in \mathcal{A};\ c>-1),$$

which shows that

$$(3.10)\qquad f \in \mathcal{R}(\gamma) \Longrightarrow \mathcal{J}_c f \in \mathcal{R}(\gamma) \quad (c>-1).$$

The assertion of Theorem 2 now follows from the observations (3.7) and (3.10).

Corollary 2. *Under the hypotheses of Theorem* 2,

$$J_{0,z}^{\alpha,-\alpha,\eta} f(z) \in \mathcal{H}^{\infty}.$$

Proof. Corollary 2 can be proven easily by applying the relationship

(3.7) and Theorem 3 of Jung *et al.* [3].

Finally, we give an interesting application of Theorem 2 involving the generalized hypergeometric function ${}_lF_m(z)$ defined by (2.1).

Theorem 3. *Let the function*

$$z\,{}_lF_m(\lambda_1, \cdots, \lambda_l; \mu_1, \cdots, \mu_m; z) \quad (l \leqq m+1)$$

be in the class $\mathcal{R}(\gamma)$.

Then

$$(3.11) \quad z\,{}_{l+s}F_{m+s}(\lambda_1, \cdots, \lambda_l, 2, \cdots, 2; \mu_1, \cdots, \mu_m, \alpha_1+2, \cdots, \alpha_s+2; z) \in \mathcal{R}(\gamma) \quad [\alpha_j \in N\ (j=1, \cdots, s)].$$

Proof. The assertion (3.11) follows, in view of (3.7) and (3.8), when we make an iterative use of Theorem 2.

A similar use of Corollary 2 yields

Corollary 3. *Under the hypothesis of Theorem* 3,

$$(3.12) \quad z\,{}_{l+s}F_{m+s}(\lambda_1, \cdots, \lambda_l, 2, \cdots 2; \mu_1, \cdots, \mu_m, \alpha_1+2, \cdots, \alpha_s+2; z) \in \mathcal{H}^\infty \quad [\alpha_j \in N\ (j=1, \cdots, s)].$$

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