# 77. *Weakly Compact Weighted Composition Operators on Certain Subspaces of* $C(X, E)$

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Let $X$ be a compact Hausdorff space and $E$ a complex Banach space. By $C(X, E)$ we denote the Banach space of all continuous $E$-valued functions on $X$ with the supremum norm. The compact weighted composition operators on $C(X, E)$ have been characterized by Jamison and Rajagopalan [2]. One of the authors proved an analogue for those operators on a more general space $A(X, E)$ ([6]). In this note, we characterize the weakly compact weighted composition operators on $A(X, E)$, and give some remarks on the difference between compactness and weak compactness of weighted composition operators.

Let $A$ be a function algebra on $X$, that is, a uniformly closed subalgebra of $C(X)=C(X, \boldsymbol{C})$ which contains the constants and separates the points of $X$. We define the closed subspace $A(X, E)$ of $C(X, E)$ by

$$A(X, E)=\{f \in C(X, E) : e^* \circ f \in A \text{ for all } e^* \in E^*\},$$

where $E^*$ is the dual space of $E$. We recall that a *weighted composition operator* of $A(X, E)$ is a bounded linear operator $T$ from $A(X, E)$ into itself, which has the form;

$$Tf(x)=w(x)f(\varphi(x)), \qquad x \in X,\ f \in A(X, E),$$

for some selfmap $\varphi$ of $X$ and some map $w$ from $X$ into $B(E)$, the space of bounded linear operators on $E$. In the sequel, we write $wC_\varphi$ in place of $T$. For a weighted composition operator $wC_\varphi$ on $A(X, E)$, we have that $|||w|||=\sup\{\|w(x)\|_{B(E)} : x \in X\}<+\infty$, and that the map $w : X \to B(E)$ is continuous in the strong operator topology. We also know that $\varphi$ is continuous on an open set $S(w)=\{x \in X : w(x) \neq O\}$ in $X$. Since $X$ is imbedded into the maximal ideal space $M_A$ of $A$, we sometimes consider the selfmap $\varphi$ of $X$ as a map from $X$ into $M_A$. Notice that $M_A$ is decomposed into (Gleason) parts for $A$. If every non-trivial part $P$ satisfies the condition below, then the associated space $A(X, E)$ is said to have the *property* $(\alpha)$;

> *for any* $x \in P$, *there are an open neighborhood* $V$ *of* $x$ *relative to* $P$ *and a homeomorphism* $\rho$ *from a polydisc* $D^N$ *(*$N$ *depends on* $x$*) onto* $V$ *such that* $\hat{f} \circ \rho$ *is analytic on* $D^N$ *for the Gelfand transform* $\hat{f}$ *of any* $f \in A$ (see [3]).

Simple examples of $A(X, E)$ with the property $(\alpha)$ are $C(X, E)$ and $\{f \in C(\bar{D}, E) : f$ is an analytic E-valued function on the interior of $\bar{D}\}$, where $\bar{D}$ is the closed unit disc. As a matter of notational convenience, we put $E_0=\{e \in E : \|e\|_E \leq 1\}$, and $E_0^*=\{e^* \in E^* : \|e^*\| \leq 1\}$. In what follows, we understand that $E_0^*$ is given the weak* topology.

**Theorem 1.** *Let $wC_\varphi$ be a weighted composition operator on $A(X,E)$.*

(a) *If $wC_\varphi$ is weakly compact, then*

(i) *for each connected component $C$ of $S(w)$, there exist an open set $U$ containing $C$ and a part $P$ for $A$ such that $\varphi(U)\subset P$;*

(ii) *when $\{x_\lambda\}$ is a net in $X$ converging to $x_0$ and $\{e_\lambda^*\}$ is a net in $E_0^*$ converging to $e_0^*$ in the weak* topology, given $\varepsilon>0$ and $\lambda_0$, there exists a finite set of indices $\lambda_i\geq\lambda_0$, $i=1,\cdots,k$, such that for each $e\in E_0$,*

$$\min_{1\leq i\leq k}|e_{\lambda_i}^*(w(x_{\lambda_i})e)-e_0^*(w(x_0)e)|<\varepsilon. \tag{1}$$

(b) *In addition, we assume that $A(X,E)$ has the property $(\alpha)$, and that $S(w)=X$. If $wC_\varphi$ satisfies the above conditions* (i) *and* (ii), *then $wC_\varphi$ is weakly compact.*

For the proof, we need the lemma:

**Lemma.** *A bounded subset $F$ of $A(X,E)$ is weakly relatively compact if and only if the following holds: If $\{x_\lambda\}$ is a net in $X$ converging to $x_0$ and $\{e_\lambda^*\}$ is a net in $E_0^*$ converging to $e_0^*$ in the weak* topology, given $\varepsilon>0$ and $\lambda_0$, there exists a finite set of indices $\lambda_i\geq\lambda_0$, $i=1,\cdots,k$, such that for each $f\in F$,*

$$\min_{1\leq i\leq k}|e_{\lambda_i}^*(f(x_{\lambda_i}))-e_0^*(f(x_0))|<\varepsilon.$$

*Proof.* For each $f\in A(X,E)$, define a continuous function $\tilde{f}$ on the product space $X\times E_0^*$ by $\tilde{f}(x,e^*)=e^*(f(x))$ for each $(x,e^*)\in X\in E_0^*$. It is easily seen that the map $\Phi: f\to\tilde{f}$ is an embedding of $A(X,E)$ into $C(X\times E_0^*)$. So, $F$ is weakly relatively compact in $A(X,E)$ if and only if $\Phi(F)$ is weakly relatively compact in $C(X\times E_0^*)$. The lemma follows from [1, Theorem IV.6.14].

*Proof of Theorem* 1. (a) Suppose that $wC_\varphi$ is weakly compact. As in the proof of [6, Theorem], we show the condition (i) by inducing the contradiction from the assumption that there exists a point $x_0$ in $S(w)$ such that $\varphi(U)\not\subset P_0$ for any neighborhood $U$ of $x_0$, where $P_0$ is the part containing $\varphi(x_0)$. Take $e_0\in E$ with $\delta=\|w(x_0)e_0\|>0$, and put $U_0=\{x\in X:\|w(x)e_0\|>\frac{3}{4}\delta\}$. Using the above assumption, we can inductively construct a sequence $\{x_n\}$ in $U_0$ and a norm 1 sequence $\{F_n\}$ in $A$ such that $F_n(\varphi(x_m))=0$ $(1\leq m\leq n)$, and $|F_l(\varphi(x_n))|>\frac{1}{2}$ $(1\leq l<n)$, for all $n=1,2,\cdots$. Without loss of generality, we may assume that $\{x_n\}$ is converging to some point $x_\infty$ in $X$. Since $\|w(x_\infty)e_0\|_E\geq\frac{3}{4}\delta$, we can find $e_0^*\in E_0^*$ with $|e_0^*(w(x_\infty)e_0)|\geq\frac{3}{4}\delta$. Set $f_n(x)=F_n(x)e_0$, for all $x\in X$, and $n=1,2,\cdots$. Then $\{wC_\varphi f_n\}$ is weakly relatively compact, and so the lemma enables us to find a finite set $\{x_{n_1},\cdots,x_{n_k}\}$ such that for each $n=1,2,\cdots$,

$$\min_{1\leq i\leq k}|e_0^*(wC_\varphi f_n(x_{n_i}))-e_0^*(wC_\varphi f_n(x_\infty))|<\frac{\delta}{4}. \tag{2}$$

Let $n_0=\max\{n_1,\cdots,n_k\}$, and select $m_0$ $(m_0>n_0)$ such that $|F_{n_0}(\varphi(x_{m_0}))-F_{n_0}(\varphi(x_\infty))|<\frac{1}{6}$. It follows $|F_{n_0}(\varphi(x_\infty))|>|F_{n_0}(\varphi(x_{m_0}))|-\frac{1}{6}>\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$, while we have $F_{n_0}(\varphi(x_{n_i}))=0$, for all $i=1,2,\cdots,k$. Hence, for each $i$, we have

$$\begin{aligned}&|e_0^*(wC_\varphi f_{n_0}(x_{n_i}))-e_0^*(wC_\varphi f_{n_0}(x_\infty))|\\&=|e_0^*(w(x_{n_i})F_{n_0}(\varphi(x_{n_i}))-w(x_\infty)F_{n_0}(\varphi(x_\infty)))e_0|\\&=|F_{n_0}(\varphi(x_\infty))||e_0^*(w(x_\infty))e_0|\geq\frac{1}{3}\cdot\frac{3}{4}\delta=\frac{1}{4}\delta,\end{aligned}$$

which contradicts (2).

We next show the condition (ii). For each $e\in E_0$, put $f_e(x)=e$ for all $x\in X$. Then $\{wC_\varphi f_e : e\in E_0\}$ is weakly relatively compact. Applying the lemma, we obtain (ii), and the part (a) is proved.

(b) Suppose that $A(X,E)$ has $(\alpha)$, and that $S(w)=X$. Using (i) and (ii), we must show that $wC_\varphi$ is weakly compact. For this purpose, let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in $X$ with $x_\lambda\to x_0$, and $\{e_\lambda^*\}_{\lambda\in\Lambda}$ a net in $E_0^*$ with $e_\lambda^*\xrightarrow{w^*}e_0^*$, while $\varepsilon>0$ and $\lambda_0$ given. By (i) there are an open neighborhood $U$ of $x_0$ and a part $P$ such that $\varphi(U)\subset P$. We here assume that $P$ is non-trivial (if $P$ is one-point, our consideration will be more simple). Then we have an open neighborhood $V$ of $\varphi(x_0)$ in $P$ and a homeomorphism $\rho$ from $D^N$ onto $V$ described in the property $(\alpha)$. Put $A_0(X,E)=\{f\in A(X,E) : \|f\|\leq 1\}$. The set $\{(e_\lambda^*\circ w(x_\lambda)\circ f)^\wedge\circ\rho : f\in A_0(X,E), \lambda\in\Lambda\}$ consists of analytic functions on $D^N$ bounded by $|||w|||$, and so the Montel theorem presents us an open neighborhood $W$ of $z_0=\rho^{-1}(\varphi(x_0))$ in $D^N$ such that

$$\begin{aligned}(3)\qquad&|e_\lambda^*(w(x_\lambda)f(\varphi(x)))-e_\lambda^*(w(x_\lambda)f(\varphi(x_0)))|\\&=|((e_\lambda^*\circ w(x_\lambda)\circ f)^\wedge\circ\rho)(z)-((e_\lambda^*\circ w(x_\lambda)\circ f)^\wedge\circ\rho)(z_0)|<\varepsilon,\end{aligned}$$

for all $z=\rho^{-1}(\varphi(x))\in W\cap\rho^{-1}(\varphi(X))$, $f\in A_0(X,E)$, and $\lambda\in\Lambda$. Take an indice $\lambda'$ $(\lambda'\geq\lambda_0)$ such that for each $\lambda\geq\lambda'$, $x_\lambda$ belongs to an open neighborhood $U\cap\varphi^{-1}(\rho(W))$ of $x_0$. By (ii) we get a finite set of indices $\lambda_i\geq\lambda'$, $i=1,\cdots,k$, such that (1) holds for each $e\in E_0$, therefore

$$(4)\qquad\min_{1\leq i\leq k}|e_{\lambda_i}^*(w(x_{\lambda_i})f(\varphi(x_0)))-e_0^*(w(x_0)f(\varphi(x_0)))|<\varepsilon.$$

holds for each $f\in A_0(X,E)$. Since $\rho^{-1}(\varphi(x_{\lambda_i}))\in W$, (3) and (4) yield

$$\min_{1\leq i\leq k}|e_{\lambda_i}^*(w(x_{\lambda_i})f(\varphi(x_{\lambda_i})))-e_0^*(w(x_0)f(\varphi(x_0)))|<2\varepsilon,$$

for each $f\in A_0(X,E)$. It follows from the lemma that $wC_\varphi(A_0(X,E))$ is weakly relatively compact, which was to be proved.

We here take up the question; when is a weakly compact weighted composition operator compact? In [6], we had already characterized the compact ones, namely, we know that the compact case of Theorem 1 is obtained by changing the condition (ii) for two conditions; (ii)′ the map $w : X\to B(X)$ is continuous in the uniform operator topology; and (iii)′ for each $x\in S(w)$, $w(x)$ is a compact operator on $E$, and that it holds without the assumption that $S(w)=X$ in (b). To clarify the difference between compactness and weak compactness, we remark that this result can be restated as follows:

**Theorem 2.** *Let $wC_\varphi$ be a weighted composition operator on $A(X,E)$.*

(a) *If $wC_\varphi$ is compact, then*

(i) *for each connected component $C$ of $S(w)$, there exist an open* *set $U$ containing $C$ and a part $P$ for $A$ such that $\varphi(U) \subset P$;*

(ii) *when $\{x_\lambda\}$ is a net in $X$ converging to $x_0$ and $\{e_\lambda^*\}$ is a net in $E_0^*$ converging to $e_0^*$ in the weak* topology, given $\varepsilon > 0$, there exists an indice $\lambda_0$ such that $\lambda \geq \lambda_0$ implies*

$$|e_\lambda^*(w(x_\lambda)e) - e_0^*(w(x_0)e)| < \varepsilon,$$

*for each $e \in E_0$.*

(b) *In addition, we assume that $A(X, E)$ has the property* ($\alpha$). *Then the converse to the part* (a) *is true.*

Our theorems have two corollaries, the proofs of which are elementary.

**Corollary 1.** *Suppose that $A(X, E)$ has the property ($\alpha$), and let $wC_\varphi$ be a weighted composition operator on $A(X, E)$.*

(a) *If $E$ is finite dimensional, the following are equivalent;*

(i) *$wC_\varphi$ is compact;*

(ii) *$wC_\varphi$ is weakly compact;*

(iii) *for each connected component $C$ of $S(w)$, there exist an open set $U$ containing $C$ and a part $P$ for $A$ such that $\varphi(U) \subset P$.*

(b) *If $E$ is reflexive and $S(w) = X$, the above conditions* (ii) *and* (iii) *are equivalent.*

We recall that a *composition operator* on $A(X, E)$ is a weighted composition operator $wC_\varphi$ on $A(X, E)$ in the special case of $w(x) = I_E$, the identity operator of $E$, for each $x \in X$. In [6], we see that if $E$ is infinite dimensional, there is no compact composition operator on $A(X, E)$. Similarly, we have

**Corollary 2.** *If $E$ is not reflexive, there is no weakly compact composition operator on $A(X, E)$.*

Suppose that $A(X, E)$ has the property ($\alpha$), and let $C_\varphi$ be a composition operator on $A(X, E)$ induced by the selfmap $\varphi$ of $X$ such that $\varphi(X) \subset P$. If $E$ is finite dimensional, $C_\varphi$ is compact by Corollary 1 (a); if $E$ is infinite dimensional and reflexive, $C_\varphi$ is not compact by [6, Corollary 2], but is weakly compact by Corollary 1 (b); if $E$ is not reflexive, $C_\varphi$ is not even weakly compact by Corollary 2 (cf. [4]). We conclude this note with a remark that in Corollary 1 the property ($\alpha$) is necessary; there is a weighted composition operator $wC_\varphi$ on a function algebra $A(X, \mathbf{C})$ without ($\alpha$), which has (iii) but not (ii) (see [5] for the example).

## References

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