76. On the Asymptotic Remainder Estimate for the Eigenvalues of Operators Associated with Strongly Elliptic Sesquilinear Forms

By Yôichi MIYAZAKI
School of Dentistry, Nihon University

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1991)

§ 1. **Introduction and main result.** This present note is devoted to the supplementary result to be added to the previous paper [5].

Let Ω be a bounded domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. For a nonnegative integer m and $p>1$ we denote by $W_p^m(\Omega)$ with the norm $\| \ \|_{m,p}$ the space of functions whose distributional derivatives of order up to m belong to $L_p(\Omega)$, and by $W_{p,0}^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega)$. In particular we set $H^m(\Omega)=W_2^m(\Omega)$, $\| \ \|_m=\| \ \|_{m,2}$ and $H_0^m(\Omega)=W_{2,0}^m(\Omega)$. Let B be an integro-differential symmetric sesquilinear form of order m with bounded coefficients:

$$B[u, v]=\int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x)D^\alpha u(x)\overline{D^\beta v(x)}dx,$$

$$\alpha=(\alpha_1, \cdots, \alpha_n), \quad D^\alpha=(-\sqrt{-1})^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n},$$

which is coercive on $H_0^m(\Omega)$:

$$B[u,u]\geqq\delta\|u\|_m^2-C_0\|u\|_0^2, \quad \delta>0, \quad C_0\geqq 0 \quad \text{for any } u\in H_0^m(\Omega).$$

Let A be the operator associated with the variational triple $\{B, H_0^m(\Omega), L_2(\Omega)\}$. That is, $u\in H_0^m(\Omega)$ belongs to $D(A)$, the domain of A if and only if there exists $f\in L_2(\Omega)$ such that $B[u,v]=(f,v)_{L_2(\Omega)}$ for any $v\in H_0^m(\Omega)$ and we define $Au=f$. As is known, A is a self-adjoint operator and the spectrum of A consists of eigenvalues accumulating only at $+\infty$. For a real number t let $N(t;A)$ or simply $N(t)$ denote the number of eigenvalues of A not exceeding t. We put

$$a(x,\xi)=\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta},$$

$$\mu_A(x)=(2\pi)^{-n}\int_{a(x,\xi)<1} d\xi, \quad \mu_A(\Omega)=\int_\Omega \mu_A(x)dx.$$

For $\tau=k+\sigma>0$ with an integer k and $0<\sigma\leqq 1$ let $\mathscr{B}^\tau(\Omega)$ denote the space of functions u in Ω such that $D^\alpha u$ are bounded and continuous for $|\alpha|\leqq k$ and $|D^\alpha u(x)-D^\alpha u(y)|/|x-y|^\sigma$ $(x, y\in\Omega, x\neq y)$ are bounded for $|\alpha|=k$.

In [5] we investigated the remainder estimate in the asymptotic formula for the eigenvalues of A with $a_{\alpha\beta}\in\mathscr{B}^\tau(\Omega)$ $(|\alpha|=|\beta|=m)$ for $\tau>0$. But we could not give any assertion for $0<\tau<m$ when $2m\leqq n$. In this note we settle this case.

Theorem. *Let $\tau>0$. Suppose that $a_{\alpha\beta}\in\mathscr{B}^\tau(\Omega)$ $(|\alpha|=|\beta|=m)$ and that the boundary $\partial\Omega$ is in C^{2m}-class. Then we have*

$$N(t)=\mu_A(\Omega)t^{n/2m}+O(t^{(n-\theta)/2m}) \quad \textit{as } t\to\infty,$$

with $\theta=\tau/(\tau+1)$.

Remark. Theorem has already been obtained by Métivier [4] when $0<\tau\leqq 1$ and by the author when $2m>n$ ([5, Theorem 1]) or $\tau\geqq m$ ([5, Theorem 3]). In addition, when $2m>n$ or $0<\tau\leqq 1$, Theorem remains valid under much weaker conditions on the smoothness of $\partial\Omega$. Hence our result obtained in Theorem is new for the case of $2m\leqq n$ and $1<\tau<m$.

Theorem will be proved essentially along the same line as that of [5, Theorem 3] in which we have proceeded as follows: First we approximate A by operators A_ε $(\varepsilon>0)$ with smooth coefficients, and estimate the kernel of the resolvent $(A_\varepsilon-\lambda)^{-1}$ by using the L_p-theory or following the argument of Tanabe [7] which goes back to Beals [2]. Then applying Tsujimoto's theorem to a family of operators $\{A_\varepsilon\}_{\varepsilon>0}$, we get the asymptotic behavior of the spectral function of A_ε, from which we finally obtain the asymptotic formula for $N(t)$.

But in the proof of Theorem we need to change the above course a little, because $D^\gamma a^\varepsilon_{\alpha\beta}(x)$, defined below, cannot necessarily be estimated by a constant independent of ε when $0<\tau<m$. The resolvent kernel will be estimated not for $|\lambda|\geqq C$ but for $|\lambda|\geqq C\varepsilon^{-2m}$ with an appropriate constant C independent of ε. Hence we must consider $A_\varepsilon+C\varepsilon^{-2m}$ instead of A_ε when we apply Tsujimoto's theorem.

Let

$$M=\max_{|\alpha|+|\beta|\leqq 2m}\sup_{x\in\Omega}|a_{\alpha\beta}(x)|+\max_{|\alpha|=|\beta|=m}\max_{|\gamma|=k}\sup_{x,y\in\Omega,x\neq y}\frac{|D^\gamma a_{\alpha\beta}(x)-D^\gamma a_{\alpha\beta}(y)|}{|x-y|^\sigma}+\delta^{-1}.$$

In the following we denote by C [resp. C'] positive constants which may differ from each other and which depend only on n, m, Ω and M [resp. n, m, Ω, M and p]. When we distinguish these constants C [resp. C'], we write C_1, C_2, $\cdots$ [resp. $C'_1, C'_2, \cdots$].

§ 2. The estimate for the resolvent kernel. First we construct the operator A_ε approximating A. For $\tau=k+\sigma>0$ with an integer k and $0<\sigma\leqq 1$ we take a function $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ with supp $\varphi\subset\{x\in\boldsymbol{R}^n\,;\,|x|<1\}$ satisfying

$$\int_{\boldsymbol{R}^n}\varphi(x)dx=1,\qquad \int_{\boldsymbol{R}^n}x^\alpha\varphi(x)dx=0\quad(1\leqq|\alpha|\leqq k)$$

([5, Lemma 5.1]), and put $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$.

For $\varepsilon>0$ we consider the form

$$B_\varepsilon[u,v]=\int_\Omega\sum_{|\alpha|=|\beta|=m}a^\varepsilon_{\alpha\beta}(x)D^\alpha u(x)\overline{D^\beta v(x)}dx,$$

where

$$a^\varepsilon_{\alpha\beta}(x)=\varphi_\varepsilon * a_{\alpha\beta}(x).$$

Here the above convolution is well-defined, because $\mathscr{B}^\tau(\Omega)\subset\mathscr{B}^\tau(\boldsymbol{R}^n)$ ([5, Lemma 5.2]). It follows that

$$|a^\varepsilon_{\alpha\beta}(x)-a_{\alpha\beta}(x)|\leqq C\varepsilon^\tau,\quad |D^\gamma a^\varepsilon_{\alpha\beta}(x)|\leqq C\varepsilon^{-2m+|\alpha|+|\beta|-|\gamma|},$$

and that B_ε is coercive for sufficiently small ε. Let A_ε be the operator associated with the variational triple $\{B_\varepsilon,\ H_0^m(\Omega),\ L_2(\Omega)\}$.

We define $\mathcal{A}_\varepsilon$, $\mathcal{A}'_\varepsilon$ and $a^\varepsilon_\alpha(x)$ by

$$\mathcal{A}_\varepsilon=\mathcal{A}_\varepsilon(x, D)=\sum_{|\alpha|=|\beta|=m} D^\beta(a^\varepsilon_{\alpha\beta}D^\alpha\cdot)=\sum_{m\leq|\alpha|\leq 2m} a^\varepsilon_\alpha(x)D^\alpha,\quad \mathcal{A}'_\varepsilon=\sum_{|\alpha|=2m} a^\varepsilon_\alpha(x)D^\alpha.$$

For $p>1$ we define $A_{\varepsilon,p}$ by

$$D(A_{\varepsilon,p})=W^{2m}_p(\Omega)\cap W^m_{p,0}(\Omega),\quad (A_{\varepsilon,p}u)(x)=\mathcal{A}_\varepsilon(x, D)u(x)\quad \text{for } u\in D(A_{\varepsilon,p}),$$

and have $A_\varepsilon=A_{\varepsilon,2}$ from the regularity theorem.

Lemma 1. *There exist $C'_1>0$, $C'_2>0$ and $0<\varepsilon_0<1$ such that*

$$\|u\|_{2m,p}+|\lambda|\|u\|_{0,p}\leqq C'_1\|(A_{\varepsilon,p}-\lambda)u\|_{0,p}\quad \textit{for } u\in D(A_{\varepsilon,p}),$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_2\varepsilon^{-2m}$ and $|\arg(-\lambda)|\leqq 3\pi/4$.

Proof. It is known that there exist $C'_3>0$, $C'_4>0$ and $0<\varepsilon_0<1$ such that

$$(1)\qquad \|u\|_{2m,p}+|\lambda|\|u\|_{0,p}\leqq C'_3\|(\mathcal{A}'_\varepsilon-\lambda)u\|_{0,p}\quad \text{for } u\in D(A_{\varepsilon,p}),$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_4$ and $|\arg(-\lambda)|\leqq 3\pi/4$ ([1]).

Using the interpolation inequality

$$\varepsilon^j\|u\|_{j,p}\leqq C'(\gamma\varepsilon^{2m}\|u\|_{2m,p}+\gamma^{-j/(2m-j)}\|u\|_{0,p}),\quad 0\leqq j\leqq 2m-1$$

for $\varepsilon>0$ and $\gamma>0$, we have

$$(2)\qquad \begin{aligned}\|(A_{\varepsilon,p}-\mathcal{A}'_\varepsilon)u\|_{0,p}&\leqq\sum_{m\leqq|\alpha|\leqq 2m-1}\|a^\varepsilon_\alpha(x)D^\alpha u\|_{0,p}\\ &\leqq C'\sum_{j=m}^{2m-1}\varepsilon^{-2m+j}\|u\|_{j,p}\\ &\leqq C'(\gamma\|u\|_{2m,p}+\gamma^{1-2m}\varepsilon^{-2m}\|u\|_{0,p})\end{aligned}$$

for any $\gamma>0$. In view of (1) and (2) we get

$$\|u\|_{2m,p}+|\lambda|\|u\|_{0,p}\leqq C'_3\|(A_{\varepsilon,p}-\lambda)u\|_{0,p}+C'_5(\gamma\|u\|_{2m,p}+\gamma^{1-2m}\varepsilon^{-2m}\|u\|_{0,p}).$$

Taking γ so that $C'_5\gamma\leqq 1/2$ and putting $C'_2=\max\{2C'_5\gamma^{1-2m}, C'_4\}$ and $C'_1=2C'_3$, we get the lemma. Q.E.D.

For $\eta=(\eta_1,\cdots,\eta_n)\in \boldsymbol{C}^n$, we define $A^\eta_{\varepsilon,p}$ by

$$D(A^\eta_{\varepsilon,p})=D(A_{\varepsilon,p})=W^{2m}_p(\Omega)\cap W^m_{p,0}(\Omega),$$
$$(A^\eta_{\varepsilon,p}u)(x)=e^{-x\eta}\mathcal{A}_\varepsilon(x, D)(e^{x\eta}u(x))\quad \text{for } u\in D(A^\eta_{\varepsilon,p}).$$

Lemma 2. *There exist $C'_6>0$, $C'_7>0$ and $0<\varepsilon_0<1$ such that*

$$\|u\|_{2m,p}+|\lambda|\|u\|_{0,p}\leqq C'_6\|(A^\eta_{\varepsilon,p}-\lambda)u\|_{0,p}\quad \textit{for } u\in D(A^\eta_{\varepsilon,p}),$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_7|\eta|^{2m}\geqq C'_7\varepsilon^{-2m}$ and $|\arg(-\lambda)|\leqq 3\pi/4$.

Proof. When $\varepsilon^{-1}\leqq|\eta|$, we have for $\gamma>0$

$$\begin{aligned}\|(A^\eta_{\varepsilon,p}-A_{\varepsilon,p})u\|_{0,p}&\leqq\sum_{m\leqq|\alpha|\leqq 2m}\|a^\varepsilon_\alpha(x)\{(D-i\eta)^\alpha-D^\alpha\}u\|_{0,p}\\ &\leqq C'\sum_{k=m}^{2m}\varepsilon^{-2m+k}\sum_{j=1}^{k}|\eta|^j\|u\|_{k-j,p}\\ &\leqq C'(\gamma\|u\|_{2m,p}+\gamma^{1-2m}|\eta|^{2m}\|u\|_{0,p}).\end{aligned}$$

This combined with Lemma 1 gives

$$\|u\|_{2m,p}+|\lambda|\|u\|_{0,p}\leqq C'_1\|(A^\eta_{\varepsilon,p}-\lambda)u\|_{0,p}+C'_8(\gamma\|u\|_{2m,p}+\gamma^{1-2m}|\eta|^{2m}\|u\|_{0,p}),$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_2\varepsilon^{-2m}$, $|\arg(-\lambda)|\leqq 3\pi/4$ and $\varepsilon^{-1}\leqq|\eta|$. Taking γ so that $C'_8\gamma\leqq 1/2$ and putting $C'_7=\max\{2C'_8\gamma^{1-2m}, C'_2\}$ and $C'_6=2C'_1$, we get the lemma. Q.E.D.

Lemma 3. *There exist $C'_9>0$, $C'_{10}>0$ and $0<\varepsilon_0<1$ such that $\lambda\in\rho(A^\eta_{\varepsilon,p})$, the resolvent set of $A^\eta_{\varepsilon,p}$ and*

$$\|(A^\eta_{\varepsilon,p}-\lambda)^{-1}f\|_{0,p}\leqq C'_9|\lambda|^{-1}\|f\|_{0,p},\quad \|(A^\eta_{\varepsilon,p}-\lambda)^{-1}f\|_{2m,p}\leqq C'_9\|f\|_{0,p}$$

for $f\in L_p(\Omega)$, when $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_{10}|\eta|^{2m}\geqq C'_{10}\varepsilon^{-2m}$ and $|\arg(-\lambda)|\leqq 3\pi/4$.

Proof. Since Lemma 2 shows that $A^\eta_{\varepsilon,p}-\lambda$ is one-to-one, it remains to prove that $A^{\eta}_{\varepsilon,p}-\lambda$ is onto. For this proof we follow the argument of Tanabe [6, pp. 84–87]. We map Ω into a C^{∞}-domain $\tilde{\Omega}$ by a C^{2m}-diffeomorphism, which transform $A^{\eta}_{\varepsilon,p}$ into $\tilde{A}^{\eta}_{\varepsilon,p}$ with continuous coefficients. We approximate $\tilde{A}^{\eta}_{\varepsilon,p}$ by $\tilde{A}^{\eta}_{\varepsilon,\gamma,p}=\varphi_{\gamma}*\tilde{A}^{\eta}_{\varepsilon,p}$ $(\gamma>0)$ with C^{∞}-coefficients. From Lemma 2 it follows that there exist $C'_{11}>0$, $C'_{12}>0$ and $0<\gamma_0<1$ such that

$$\|v\|_{2m,p}+|\lambda|\|v\|_{0,p}\leqq C'_{11}\|(\tilde{A}^{\eta}_{\varepsilon,\gamma,p}-\lambda)v\|_{0,p}\quad \text{for } v\in D(\tilde{A}^{\eta}_{\varepsilon,\gamma,p}),$$

when $0<\gamma<\gamma_0$, $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C'_{12}|\eta|^{2m}\geqq C'_{12}\varepsilon^{-2m}$ and $|\arg(-\lambda)|\leqq 3\pi/4$. Further for the formally adjoint operator $(\tilde{A}^{\eta}_{\varepsilon,\gamma,p})^{*}$ and q with $p^{-1}+q^{-1}=1$ there exists $C_{\gamma}>0$, which may depend on γ, ε and η, such that

$$\|v\|_{2m,q}+|\lambda|\|v\|_{0,q}\leqq 2C'_{11}\|((\tilde{A}^{\eta}_{\varepsilon,\gamma,p})^{*}-\lambda)v\|_{0,q}\quad \text{for } v\in W^{2m}_{q}(\tilde{\Omega})\cap W^{m}_{q,0}(\tilde{\Omega}),$$

when $0<\gamma<\gamma_0$, $0<\varepsilon<\varepsilon_0$, $|\lambda|\geqq C_{\gamma}$ and $|\arg(-\lambda)|\leqq 3\pi/4$. Then applying Schechter's result we see that $\lambda\in\rho(\tilde{A}^{\eta}_{\varepsilon,\gamma,p})$ for $|\lambda|\geqq\max\{C'_{12}|\eta|^{2m},C_{\gamma}\}$. Since $\lambda\in\rho(\tilde{A}^{\eta}_{\varepsilon,\gamma,p})$ and $|\mu-\lambda|\|(\tilde{A}^{\eta}_{\varepsilon,\gamma,p}-\lambda)^{-1}\|<1$ imply $\mu\in\rho(\tilde{A}^{\eta}_{\varepsilon,\gamma,p})$, it follows that $\lambda\in\rho(\tilde{A}^{\eta}_{\varepsilon,\gamma,p})$ if $|\lambda|\geqq C'_{12}|\eta|^{2m}\geqq C'_{12}\varepsilon^{-2m}$. From this we conclude that $A^{\eta}_{\varepsilon,p}-\lambda$ is onto. Hence the lemma follows. Q.E.D.

From the embedding theorem, the integral kernel theorem and Lemma 3 it follows that there exist an integer k and a_j, $0<a_j<1$ $(j=1,2,\cdots,k)$, determined by n and m, such that $\sum_{j=1}^{k}a_j=n/2m$ and that $\prod_{j=1}^{k}(A^{\eta}_{\varepsilon,2}-\lambda_j)^{-1}$ has a continuous kernel satisfying

$$(3)\qquad \left|\mathcal{K}\left[\prod_{j=1}^{k}(A^{\eta}_{\varepsilon,2}-\lambda_j)^{-1}\right](x,y)\right|\leqq C_1\prod_{j=1}^{k}|\lambda_j|^{a_j-1},$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda_j|\geqq C_2|\eta|^{2m}\geqq C_2\varepsilon^{-2m}$ and $|\arg(-\lambda_j)|\leqq 3\pi/4$ $(1\leqq j\leqq k)$ (the detail discussion is found in [7]). Here and in the following we denote by $\mathcal{K}[T](x,y)$ the kernel of an integral operator T. Combining (3) with

$$\mathcal{K}\left[\prod_{j=1}^{k}(A_{\varepsilon}-\lambda_j)^{-1}\right](x,y)=e^{(x-y)\eta}\mathcal{K}\left[\prod_{j=1}^{k}(A^{\eta}_{\varepsilon,2}-\lambda_j)^{-1}\right](x,y),$$

and substituting $\eta=-(x-y)(C_2^{-1}\min\{|\lambda_1|,\cdots,|\lambda_k|\})^{1/2m}/|x-y|$, we obtain

$$\left|\mathcal{K}\left[\prod_{j=1}^{k}(A_{\varepsilon}-\lambda_j)^{-1}\right](x,y)\right|\leqq C_3\sum_{h=1}^{k}\exp(-C_4|\lambda_h|^{1/2m}|x-y|)\cdot\prod_{j=1}^{k}|\lambda_j|^{a_j-1},$$

when $0<\varepsilon<\varepsilon_0$, $|\lambda_j|\geqq C_2\varepsilon^{-2m}$ and $|\arg(-\lambda_j)|\leqq 3\pi/4$ $(1\leqq j\leqq k)$, from which it follows that

$$(4)\qquad \left|\mathcal{K}\left[\prod_{j=1}^{k}(A_{\varepsilon}+C_5\varepsilon^{-2m}-\lambda_j)^{-1}\right](x,y)\right|$$
$$\leqq C_6\sum_{h=1}^{k}\exp(-C_7|\lambda_h|^{1/2m}|x-y|)\cdot\prod_{j=1}^{k}|\lambda_j|^{a_j-1},$$

when $0<\varepsilon<\varepsilon_0$ and $|\arg(-\lambda_j)|\leqq 3\pi/4$ $(1\leqq j\leqq k)$ if we take $C_5=2C_2$. Note that the conditions $|\lambda_j|\geqq C_2\varepsilon^{-2m}$ $(1\leqq j\leqq k)$ have been eliminated.

The calculation in [7, pp. 275–281] leads us from (4) to the following estimate for the kernel of $(A_{\varepsilon}+C_5\varepsilon^{-2m}-\lambda)^{-1}$ through the estimate for the kernel of $\exp(-t(A_{\varepsilon}+C_5\varepsilon^{-2m}))$ for $t>0$.

Lemma 4. *There exist $C_8>0$, $C_9>0$ and $0<\varepsilon_0<1$ such that*

$$|\mathcal{K}[(A_{\varepsilon}+C_5\varepsilon^{-2m}-\lambda)^{-1}](x,y)|$$
$$\leqq\begin{cases} C_8|\lambda|^{n/2m-1}\exp(-C_9|\lambda|^{1/2m}|x-y|) & (2m>n)\\ C_8\{1+\log^{+}(|\lambda|^{1/2m}|x-y|)^{-1}\}\exp(-C_9|\lambda|^{1/2m}|x-y|) & (2m=n)\\ C_8|x-y|^{2m-n}\exp(-C_9|\lambda|^{1/2m}|x-y|) & (2m<n)\end{cases}$$

for $0<\varepsilon<\varepsilon_0$, $x, y\in\Omega$ *and* $\lambda<0$.

§3. Proof of Theorem. Now that we have attained Lemma 4, we can apply Tsujimoto's theorem ([9], see also [5, Remark of Theorem 4]) to a family of operators $\{A_\varepsilon+C_5\varepsilon^{-2m}\}_{0<\varepsilon<\varepsilon_0}$ and get for $t>1$

$$|e_\varepsilon(t-C_5\varepsilon^{-2m}; x, x)-\mu_{A_\varepsilon}(x)t^{n/2m}|\leqq C(\varepsilon\wedge\delta(x))^{-1}t^{(n-1)/2m},$$
$$0\leqq e_\varepsilon(t-C_5\varepsilon^{-2m}; x, x)\leqq Ct^{n/2m},$$

where $e_\varepsilon(t; x, y)$ is the spectral function of A_ε, $\delta(x)=\mathrm{dist}(x, \partial\Omega)$ and $\varepsilon\wedge\delta(x)=\min\{\varepsilon, \delta(x)\}$. Then it follows that

$$(5)\quad \begin{aligned}|N(t; A_\varepsilon+C_5\varepsilon^{-2m})-\mu_{A_\varepsilon}(\Omega)t^{n/2m}| &\leqq\int_\Omega |e_\varepsilon(t-C_5\varepsilon^{-2m}; x, x)-\mu_{A_\varepsilon}(x)t^{n/2m}|dx\\ &\leqq\int_{\Omega\backslash\Gamma_\varepsilon} C\varepsilon^{-1}t^{(n-1)/2m}dx+\int_{\Gamma_\varepsilon} Ct^{n/2m}dx\\ &\leqq C\varepsilon^{-1}t^{(n-1)/2m}+C\varepsilon t^{n/2m},\end{aligned}$$

where $\Gamma_\varepsilon=\{x\in\Omega; \mathrm{dist}(x, \partial\Omega)<\varepsilon\}$. Using

$$|B_\varepsilon[u, u]+C_5\varepsilon^{-2m}\|u\|_0^2-B[u, u]|\leqq C_{10}\varepsilon^\tau B[u, u]+C_{11}\varepsilon^{-2m}\|u\|_0^2,$$

and the properties of $N(t; A)$ or $N(t; B, H_0^m(\Omega), L_2(\Omega))$ ([4]), we have

$$(6)\quad N(t; A)\leqq N((1+C_{10}\varepsilon^\tau)t+C_{11}\varepsilon^{-2m}; A_\varepsilon+C_5\varepsilon^{-2m}).$$

Combining (5) and (6), and putting $\varepsilon=t^{-1/\{2m(\tau+1)\}}$, we get

$$N(t; A)-\mu_A(\Omega)t^{n/2m}\leqq C\varepsilon^\tau t^{n/2m}+C\varepsilon^{-1}t^{(n-1)/2m}\leqq Ct^{(n-\theta)/2m}$$

with $\theta=\tau/(\tau+1)$ for sufficiently large t. In the same way we get the estimate from below. Hence Theorem follows.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math., **12**, 623–727 (1959).

[2] R. Beals: Asymptotic behavior of the Green's function and spectral function of an elliptic operator. J. Funct. Anal., **5**, 484–503 (1970).

[3] F. E. Browder: On the spectral theory of elliptic differential operators. Math. Ann., **142**, 22–130 (1961).

[4] G. Métivier: Valeurs propres des problèmes aux limites elliptiques irréguliers. Bull. Soc. Math. France Mem., **51-52**, 125–219 (1977).

[5] Y. Miyazaki: A sharp asymptotic remainder estimate for the eigenvalues of operators associated with strongly elliptic sesquilinear forms. Japan. J. Math., **15**, 65–97 (1989).

[6] H. Tanabe: Equations of Evolution. Pitman, London (1979).

[7] ——: Functional Analysis. I, II. Jikkyou Shuppan, Tokyo (1980) (in Japanese).

[8] J. Tsujimoto: On the asymptotic behavior of spectral functions of elliptic operators. Japan. J. Math., **8**, 177–210 (1982).

[9] ——: Asymptotic estimates for spectral functions of elliptic operators and its application. RIMS, Kôkyûroku, **530**, 17–23 (1984) (in Japanese).