74. On Certain Multivalent Functions*)

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1. **Introduction.** Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z : |z|<1\}$.

Further, we define a function $F(z)$ by

$$F(z)=(1-\lambda)f(z)+\lambda z f'(z) \tag{1.2}$$

for $\lambda \in C$ and $f(z) \in A(p)$.

We cite the following well-known definition of convex functions in the unit disk U (cf. [5]). Suppose that $f(z)$ is analytic in U. Then the function $f(z)$ with $f'(0)\neq 0$ is said to be *convex* if and only if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad (z \in U). \tag{1.3}$$

We denote by K the subclass of $A=A(1)$ consisting of functions which are convex in U.

Let the functions $f(z)$ and $g(z)$ be analytic in U. Then the $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $w(z)$ analytic in U, with $w(0)=0$ and $|w(z)|<1$ $(z \in U)$, such that $f(z)=g(w(z))$ $(z \in U)$. We denote this subordination by $f(z)\prec g(z)$.

In [7], Saitoh proved the following theorems.

Theorem A. *Let a function* $f(z)$ *defined by* (1.1) *be in the class* $A(p)$. *If*

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha\left(0\leqq\alpha<\frac{p\,!}{(p-j)\,!}\,;\, z \in U\right),$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>\frac{1}{(p-j+1)\,!}\,\frac{(p-j+1)\,!\,2\alpha+p\,!}{2(p-j+1)+1} \quad (z \in U),$$

where $1\leqq j\leqq p$.

Theorem B. *Let a function* $F(z)$ *be defined by* (1.2) *for* $\lambda>0$ *and* $f(z)\in A(p)$. *If*

$$\mathrm{Re}\left\{\frac{F^{(j)}(z)}{z^{p-j}}\right\}>\alpha \quad \left(0\leqq\alpha<\frac{p\,!(1-\lambda+p\lambda)}{(p-j)\,!}\,;\, z \in U\right),$$

then

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\frac{(p-j)\,!\,2\alpha+p\,!\,\lambda}{(p-j)\,!\,(2-\lambda+2p\lambda)} \quad (z \in U),$$

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where $0 \leqq j \leqq p$.

These estimates are not sharp. In this paper, we give sharp results for above theorems.

2. Main results. New, we prove the following theorem.

Theorem 1. *Let a function* $f(z)$ *defined by* (1.1) *be in the class* $A(p)$. *If*

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>\alpha \quad \left(0\leqq\alpha<\frac{p\,!}{(p-j)\,!}\,;\, z\in U\right), \tag{2.1}$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}>\frac{2\alpha-q}{p-j+1}+2(q-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{p-j+k} \quad (z\in U), \tag{2.2}$$

where $1\leqq j\leqq p$, $q=p\,!/(p-j)\,!$.

Proof. It follows from (2.1) that $\frac{f^{(j)}(z)}{z^{p-j}}\prec h(z)$ for $f(z)\in A(p)$, where $h(z)=\frac{q+(q-2\alpha)z}{1-z}$. Then we have

$$\frac{f^{(j-1)}(z)}{z^{p-j+1}}=\frac{1}{z^{p-j+1}}\int_0^z t^{p-j}\,h(t)dt$$
$$=\frac{1}{r^{p-j+1}}\int_0^r \rho^{p-j}\,h(\rho e^{i\theta})d\rho \ (z=re^{i\theta}, t=\rho e^{i\theta}).$$

Therefore,

$$\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\}=\frac{1}{r^{p-j+1}}\int_0^r\rho^{p-j}\,\mathrm{Re}h(\rho e^{i\theta})d\rho\geqq\frac{1}{r^{p-j+1}}\int_0^r\rho^{p-j}\frac{q+(2\alpha-q)\rho}{1+\rho}d\rho$$
$$=\frac{2\alpha-q}{p-j+1}+2(q-\alpha)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{p-j+k}r^{k-1}$$
$$>\frac{2\alpha-q}{p-j+1}+2(q-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{p-j+k},$$

which completes the proof of Theorem 1.

Taking $j=p$ in Theorem 1, we have

Corollary 1. [8] *If*

$$\mathrm{Re}\{f^{(p)}(z)\}>\alpha \quad (0\leqq\alpha<p\,!\,;\, z\in U),$$

then we have

$$\mathrm{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\}>2\alpha-p\,!+2(p\,!-\alpha)\log 2 \quad (z\in U).$$

Letting $j=1$ in Theorem 1, we have

Corollary 2. *If*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\alpha \quad (0\leqq\alpha<p\,;\, z\in U),$$

then we have

$$\mathrm{Re}\left\{\frac{f(z)}{z^{p}}\right\}>\frac{2\alpha-p}{p}+2(p-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{p-1+k} \quad (z\in U).$$

Making $j=p=1$ in Theorem 1, we have

Corollary 3. *If*

$$\mathrm{Re}\{f'(z)\}>\alpha \quad (0\leqq\alpha<1\,;\, z\in U),$$

then we have

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\}>2\alpha-1+2(1-\alpha)\log 2 \quad (z\in U).$$

Corollary 3 is a well-known result (cf. [3], [4]). In order to obtain the next result, we need the following lemma due to Eenigenburg, Miller, Mocanu and Reade [1].

Lemma. *Let β and γ be complex numbers, and let $h(z)$ be convex in U, with $h(0)=c$ and $\mathrm{Re}(\beta h(z)+\gamma)>0$ $(z\in U)$. Let $p(z)=c+p_1z+p_2z^2+\cdots$ be analytic in U and let it satisfy the differential subordination*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z). \tag{2.3}$$

If the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z), \quad q(0)=c \tag{2.4}$$

has a univalent solution $q(z)$, then

$$p(z)\prec q(z)\prec h(z)$$

and $q(z)$ is the best dominant of the differential subordination (2.3).

We note that the univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2.3) if $p(z)\prec q(z)$ for all $p(z)$ which satisfy the differential subordination (2.3). If $\tilde{q}(z)$ is a dominant of (2.3) and $\tilde{q}(z)\prec q(z)$ for all dominants $q(z)$ of (2.3), the $\tilde{q}(z)$ is said to be the *best dominant* of the differential subordination (2.3). We can find more about differential subordinations in [2].

With the aid of the above lemma, we derive

Theorem 2. *Let a function $F(z)$ be defined by* (1.2) *for $\lambda\in C$ with $\mathrm{Re}(\lambda)>0$, for $f(z)\in A(p)$. If*

$$\frac{F^{(j)}(z)}{(1-\lambda+\lambda p)z^{p-j}}\prec\frac{q+(q-2\alpha)z}{1-z}=h(z)\ \left(0\leqq\alpha<q=\frac{p!}{(p-j)!}\ ;\ z\in U\right), \tag{2.5}$$

then

$$\frac{f^{(j)}(z)}{z^{p-j}}\prec q(z), \tag{2.6}$$

where

$$q(z)=q+\frac{2(p-\alpha)(1-\lambda+\lambda p)}{\lambda z^{(1-\lambda+\lambda p)/\lambda}}\cdot\int_0^z\frac{t^{(1-\lambda+\lambda p)/\lambda}}{1-t}dt, \tag{2.7}$$

$0\leqq j\leqq p$ and $q(z)$ is the best dominant.

Proof. In lemma, we choose

$$\beta=0,\quad \gamma=\frac{1-\lambda+\lambda p}{\lambda},\quad h(z)=\frac{q+(q-2\alpha)z}{1-z}.$$

Then the function $h(z)$ is convex in U with $h(0)=q$. Further, for such function $h(z)$, the differential equation has the form

$$q(z)+\frac{\lambda}{1-\lambda+\lambda p}\cdot zq'(z)=\frac{q+(q-2\alpha)z}{1-z}. \tag{2.8}$$

It follows that the equation (2.8) has the solution

$$(2.9)\qquad q(z)=\frac{1-\lambda+\lambda p}{\lambda z^{(1-\lambda+\lambda p)/\lambda}}\cdot\int_0^z t^{(1-\lambda+\lambda p)/\lambda-1}\cdot h(t)dt$$
$$=q+\frac{2(q-\alpha)(1-\lambda+\lambda p)}{\lambda z^{(1-\lambda+\lambda p)/\lambda}}\cdot\int_0^z \frac{t^{(1-\lambda+\lambda p)/\lambda}}{1-t}dt,$$

which is also convex in U (the proof is similar to the proof in the class K, see for example [6], Theorem 5), hence univalent in U with $q(0)=q$. By applying lemma, we have if $p(z)$ is analytic in U with $p(0)=q$ and if

$$(2.10)\qquad p(z)+\frac{\lambda}{1-\lambda+\lambda p}\cdot zp'(z)\prec h(z),$$

then

$$p(z)\prec q(z),$$

where $h(z)$ and $q(z)$ are defined in (2.5) and (2.8), respectively, any $q(z)$ is the best dominant of the differential subordination (2.10).

Letting

$$p(z)=\frac{f^{(j)}(z)}{z^{p-j}},$$

we have

$$p(z)+\frac{\lambda}{1-\lambda+\lambda p}\cdot zp'(z)=\frac{F^{(j)}(z)}{(1-\lambda+\lambda p)z^{p-j}}.$$

Therefore, we conclude that if

$$\frac{F^{(j)}(z)}{(1-\lambda+\lambda p)z^{p-j}}\prec\frac{q+(q-2\alpha)z}{1-z}=h(z),$$

then we have

$$\frac{f^{(j)}(z)}{z^{p-j}}\prec q(z)=q+\frac{2(q-\alpha)(1-\lambda+\lambda p)}{\lambda z^{(1-\lambda+\lambda p)/\lambda}}\cdot\int_0^z\frac{t^{(1-\lambda+\lambda p)/\lambda}}{1-t}dt.$$

Thus we complete the proof of Theorem 2.

Letting $p=1$, $j=1$ and $\lambda=1/2$ in Theorem 2, we have

Corollary 4. [4] *Let $f(z)\in A$ and $\alpha<1$. If*

$$(2.11)\qquad f'(z)+\frac{1}{2}zf''(z)\prec\frac{1+(1-2\alpha)z}{1-z},$$

then we have

$$(2.12)\qquad f'(z)\prec 2\alpha-1-4(1-\alpha)\frac{z+\log(1-z)}{z^2}$$

and the right hand side of (2.12) *is the best dominant.*

Next, Theorem 2 leads to

Theorem 3. *If the function $F(z)$ is defined by* (1.2) *with $\lambda>0$, for $f(z)\in A(p)$. If*

$$\mathrm{Re}\left\{\frac{F^{(j)}(z)}{(1-\lambda+\lambda p)z^{p-j}}\right\}>\alpha\quad\left(0\leqq\alpha<q=\frac{p!}{(p-j)!};\ z\in U\right),$$

then

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}>q+2(q-\alpha)\sum_{k=1}^{\infty}(-1)^k\frac{\gamma}{\gamma+k},$$

where

$$\gamma=\frac{1-\lambda+\lambda p}{\lambda}, \quad 0\leqq j\leqq p.$$

This estimate is sharp.

Taking $j=0$ in Theorem 3, we have

Corollary 5. *If*

$$\operatorname{Re}\left\{\frac{F(z)}{(1-\lambda+\lambda p)z^p}\right\}>\alpha \quad (0\leqq\alpha<1\,;\, z\in U),$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}>1+2(1-\alpha)\sum_{k=1}^{\infty}(-1)^k\frac{\gamma}{\gamma+k}.$$

Putting $j=p$ in Theorem 3, we have

Corollary 6. *If*

$$\operatorname{Re}\{F^{(p)}(z)\}>(1-\lambda+\lambda p)\alpha \quad (0\leqq\alpha<p\,!\,;\, z\in U),$$

then

$$\operatorname{Re}\{f^{(p)}(z)\}>p\,!+2(p\,!-\alpha)\sum_{k=1}^{\infty}(-1)^k\frac{\gamma}{\gamma+k}.$$

Letting $j=1$ in Theorem 3, we have

Corollary 7. *If*

$$\operatorname{Re}\left\{\frac{F'(z)}{(1-\lambda+\lambda p)z^{p-1}}\right\}>\alpha \quad (0\leqq\alpha<p\,;\, z\in U),$$

then

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>p+2(p-\alpha)\sum_{k=1}^{\infty}(-1)^k\frac{\gamma}{\gamma+k}.$$

Making $p=1$ and $j=0$, and $p=1$ and $j=1$ in Theorem 3, we have the following corollaries.

Corollary 8. *Let $f(z)\in A$. If*

$$\operatorname{Re}\left\{\frac{F(z)}{z}\right\}>\alpha \quad (0\leqq\alpha<1\,;\, z\in U),$$

then

$$\operatorname{Re}\left\{\frac{F(z)}{z}\right\}>1+2(1-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^k}{1+\lambda k}.$$

Corollary 9. *Let $f(z)\in A$. If*

$$\operatorname{Re}\{F'(z)\}>\alpha \quad (0\leqq\alpha<1\,;\, z\in U),$$

then

$$\operatorname{Re}\{f'(z)\}>1+2(1-\alpha)\sum_{k=1}^{\infty}\frac{(-1)^k}{1+\lambda k}.$$

Remark. Putting $\lambda=1$ in Corollary 8, we have Corollary 3.

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