73. A Note on Exponents of K-groups of Rings of Algebraic Integers

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1. In this note, we construct higher K-groups of rings of algebraic integers with arbitrary large l-exponent using the technique developed by K. Komatsu in his papers [4] [5].

Let l be an odd prime number. For an algebraic number field F, by which we always mean an algebraic extension over the field of rational numbers $\boldsymbol{Q}$ of finite degree, we denote by $\mathcal{O}_F$ the ring of algebraic integers of F, by F_∞ the cyclotomic $\boldsymbol{Z}_l$-extension of F, by F_m its m-th layer i.e., the unique cyclic extension of F contained in F_∞ of degree l^m. For an abelian torsion group X and a positive integer n, define $X_n=\{x\in X \mid l^n x=0\}$ and $X_\infty=\bigcup_{n=1}^\infty X_n$. We also define the *l-exponent* of the group X to be $\exp(X)=\max\{l^n \mid X_n\neq 0\}$. Let μ be the group of roots of unity. And we choose a generator ζ_n of each μ_n with $\zeta_{n+1}^l=\zeta_n$. For each odd integer ν, let $K_{2\nu}(\mathcal{O}_F)$ be the Quillen's 2ν-th K-group. According to Quillen [6], $K_{2\nu}(\mathcal{O}_F)$ is an abelian group of finite order.

Let k be a totally real algebraic number field. For a while, we fix a non-negative integer n_0 and put

$$k^{(n_0)}=k\cdot\boldsymbol{Q}_{n_0-1},\quad K^{(n_0)}=k^{(n_0)}(\mu_1),\quad G_\infty^{(n_0)}=\mathrm{Gal}(K_\infty^{(n_0)}/k^{(n_0)}),$$
$$\Gamma^{(n_0)}=\mathrm{Gal}(K_\infty^{(n_0)}/K^{(n_0)}),\quad\text{and}\quad \Delta^{(n_0)}=\mathrm{Gal}(K_\infty^{(n_0)}/k_\infty^{(n_0)}).$$

Let $\chi\colon \Delta^{(n_0)}\to\boldsymbol{Z}_l^\times$ be the Teichmüller character i.e., a homomorphism such that $\zeta_1^\delta=\zeta_1^{\chi(\delta)}$ for all $\delta\in\Delta^{(n_0)}$ and

$$\varepsilon_i=(\sharp(\Delta^{(n_0)}))^{-1}\sum_{\delta\in\Delta^{(n_0)}}\chi(\delta)^i\delta^{-1}\in\boldsymbol{Z}_l[\Delta^{(n_0)}]$$

the canonical orthogonal idempotent for each integer i. We choose a topological generator γ of $\Gamma^{(n_0)}$ and define an l-adic integer κ by $\zeta_m^\gamma=\zeta_m^\kappa$ $(m\geq 1)$. Let $\mathcal{T}=\varprojlim_k \mu_k$ be the Tate module, which is a free $\boldsymbol{Z}_l$-module of rank 1 and on which $G_\infty^{(n_0)}$ acts in a natural way. If X is a $G_\infty^{(n_0)}$-module, which is also a $\boldsymbol{Z}_l$-module, we define, for each integer $n\geq 0$,

$$X(n)=X\otimes_{\boldsymbol{Z}_l}\mathcal{T}\otimes_{\boldsymbol{Z}_l}\cdots\otimes_{\boldsymbol{Z}_l}\mathcal{T}\quad(n\text{ times}),$$

endowed with diagonal action of $G_\infty^{(n_0)}$. We denote, as usual, by $X^{G_\infty^{(n_0)}}$ the $G_\infty^{(n_0)}$-invariant submodule of X.

We shall prove a preliminary lemma.

Lemma 1. *Let X be an l-primary $G_\infty^{(n_0)}$-module and n a non-negative*

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integer. Then the natural isomorphism of abelian groups φ of X onto $X(n)$, which is defined by

$$\varphi(x)=x\otimes 1\otimes\cdots\otimes 1$$

for each element x of X, induces $\Gamma^{(n_0)}$-isomorphism on X_t for $t=1,2,\cdots,n_0$.

Proof. For any element x of X_t, we have

$$\begin{aligned}\varphi(x)^{\gamma}&=x^{\gamma}\otimes 1^{\gamma}\otimes\cdots\otimes 1^{\gamma}=x^{\gamma}\otimes\kappa\otimes\cdots\otimes\kappa\\&=(\kappa^{n}x^{\gamma})\otimes 1\otimes\cdots\otimes 1=x^{\gamma}\otimes 1\otimes\cdots\otimes 1=\varphi(x^{\gamma}),\end{aligned}$$

because $\kappa\equiv 1 \pmod{l^{n_0}}$ by the definition of κ. This is the claim of the lemma.

We can easily observe that

$$(X^{\Gamma(n_0)})_t=X^{\Gamma(n_0)}\cap X_t,\qquad ((X(n))^{\Gamma(n_0)})_t=(X(n))^{\Gamma(n_0)}\cap\varphi(X_t).$$

Hence we obtain the following $\Gamma^{(n_0)}$-isomorphism by Lemma 1.

(1) $$(X^{\Gamma(n_0)})_t\simeq(X(n)^{\Gamma(n_0)})_t\qquad\text{for } t=1,\cdots,n_0.$$

Let $C_m^{(n_0)}$ (resp. $C_\infty^{(n_0)}$) be the l-primary part of the ideal class group of $K_m^{(n_0)}$ (resp. $K_\infty^{(n_0)}$, which is defined by $\varinjlim_m C_m^{(n_0)}$, where limit is taken with respect to the natural map induced by the lifting of ideals). From (1) we obtain

(2) $$((\varepsilon_{-\nu}C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)})_t\simeq((\varepsilon_{-\nu}C_\infty^{(n_0)})^{\Gamma(n_0)})_t\qquad\text{for } t=1,\cdots,n_0.$$

By a well-known property of cyclotomic $\boldsymbol{Z}_l$-extensions (cf. [7], Proposition 13.26.), we have the following injections.

(3) $$\varepsilon_{-\nu}C_0^{(n_0)}\to(\varepsilon_{-\nu}C_\infty^{(n_0)})^{\Gamma(n_0)}.$$

(4) $$\varepsilon_{-\nu}C_0^{(0)}\to\varepsilon_{-\nu}C_0^{(n_0)}.$$

Combining (2), (3) and (4), we have an injection

(5) $$(\varepsilon_{-\nu}C_0^{(0)})_t\to((\varepsilon_{-\nu}C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)})_t\qquad\text{for } t=1,\cdots,n_0.$$

On the other hand, by Soulé's theorem (cf. [1], p. 286), for an odd positive integer ν, there is a canonical surjective homomorphism

(6) $$K_{2\nu}(\mathcal{O}_{k(n_0)})_\infty\to(C_\infty^{(n_0)}(\nu))^{G_\infty^{(n_0)}}=(C_\infty^{(n_0)}(\nu)^{\Delta(n_0)})^{\Gamma(n_0)}=(\varepsilon_{-\nu}C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)}.$$

By (6), we have

(7) $$\exp(\varepsilon_{-\nu}(C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)})\le\exp(K_{2\nu}(\mathcal{O}_{k(n_0)})_\infty).$$

2. Notations as in the previous section. We construct K-groups with arbitrary large l-exponent using the results obtained in the previous section. More precisely, for a given natural integer m, we construct K-groups with l-exponent larger than l^m.

Let k be a totally real field. Assume that the Iwasawa μ-invariant of $K=k(\mu_1)$ is zero. (For example, if we assume that k is an abelian over the rationals, this is always valid by the theorem of B. Ferrero and L. C. Washington ([7] § 7.5).) Take an l-extension k' of k with $[k'(\mu_1)_{\infty,+}:k(\mu_1)_{\infty,+}]=l^e$ where "+" stands for the maximal totally real subfield. Let $\lambda_{\varepsilon_{-\nu}}$ (resp. $\lambda'_{\varepsilon_{-\nu}}$) be the Iwasawa λ-invariant associated with the group $\varepsilon_{-\nu}C_\infty^{(0)}$ (resp. $(\varepsilon_{-\nu}C_\infty^{(0)})'$, the corresponding object for k'). In his paper [5] (Lemma 5), K. Komatsu showed a "piece-by-piece" version of the Riemann-Hurwitz formula of Y. Kida [3]. These are as follows.

(8) $$\lambda'_{\varepsilon_i}+s'-1=l^e(\lambda_{\varepsilon_i}+s-1),\quad\text{for the odd integer } i\ (i\equiv 1 \pmod{\#(\Delta^{(n_0)})}),$$

(9) $\lambda'_{\varepsilon_i}+s'=l^e(\lambda_{\varepsilon_i}+s)$ for the odd integer i ($i\not\equiv 1 \pmod{\sharp(\Delta^{(n_0)})}$),
where s (resp. s') is the number of prime ideals of k_∞ (resp. k'_∞) which is lying above the set S of tamely ramified prime ideals of k with respect to the extension k'/k.

If we assume that the set S contains at least two elements, then we have $\lambda'_{\varepsilon_{-\nu}}>0$ by (8) and (9). Moreover the μ-invariant for $k'(\mu_1)$ is also zero by the theorem of Iwasawa [2]. Hence replacing k by k', we may assume $\lambda_{\varepsilon_{-\nu}}>0$. Therefore the order of $\varepsilon_{-\nu}C_n^{(0)}$ is unbounded as n goes to infinity. But its rank is bounded because $\mu=0$. Hence its l-exponent is unbounded. Now choose n_0 so that it is larger than m. By taking sufficiently large n and replacing $K_0^{(0)}=k(\mu_1)$ by the n-th layer of its cyclotomic $\boldsymbol{Z}_l$-extension, we have

$$(\varepsilon_{-\nu}C_0^{(0)})_t\neq 0 \qquad \text{for } t=1,\cdots,n_0.$$

Then it follows from (5) that

$$((\varepsilon_{-\nu}C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)})_t\neq 0 \qquad \text{for } t=1,\cdots,n_0.$$

By (7), we finally obtain

$$\exp(K_{2\nu}(\mathcal{O}_{k^{(n_0)}})_\infty)\geq \exp(\varepsilon_{-\nu}(C_\infty^{(n_0)}(\nu))^{\Gamma(n_0)})\geq l^{n_0}\geq l^m$$

as desired.

Remark. In the above construction, we assumed $\sharp S\geq 2$. We explain that we can have this condition easily satisfied. We choose distinct prime numbers p_i ($i=1,2$) such that $p_i\equiv 1 \pmod{l}$ and that $(p_i, D_k)=1$, where D_k is the absolute discriminant of k. Let k_i be the unique cyclic extension of degree l over $\boldsymbol{Q}$ in the p_i-th cyclotomic field for each $i=1$, 2, and we put $\mathrm{Gal}(k_1\cdot k_2/k_i)=\langle\sigma_i\rangle$. Let $\tilde{k}$ be the subfield of $k_1\cdot k_2$ fixed by $\sigma_1\cdot\sigma_2$. Put $k'=\tilde{k}\cdot k$. Then it is easy to see that $[k':k]=[k'(\mu_1)_{\infty,+}:k(\mu_1)_{\infty,+}]=l$ and that $p_1, p_2\in S$. Hence the field k' satisfies the condition.

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